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ALGEBRAIC FACTORS:

How to Find them and How to Use them.

BY

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PREFACE.

THE study of Mathematics is rendered so much easier and more inviting by a thorough knowledge of Algebraic Factors, that it has always been a mystery to me why teachers devote so little time to their elucidation.

As a rule, this branch of the subject is hurried over with but a slight notice, both in books and in the school-room. Nothing could be more injudicious, since a true conception of factorial analysis enables the student to traverse, with the greatest ease, branches of Algebra which in other cases prove veritable Sloughs of Despond. To endeavour to arouse both teachers and pupils to a sense of the importance of a complete mastery of factors is the object of this little work. It first shows the most useful methods of obtaining the factors, and then proceeds to illustrate the more important uses to which they can be put in Elementary Algebra. This, it is thought, will be sufficient to show their potency, and will prepare the student for their more extensive use

in Advanced Algebra and the higher branches of Mathematics.

Numerous examples will be found distributed over the book, most of which are original. The miscellaneous questions are taken almost exclusively from papers set at the London University, College of Preceptors, Science and Art Department, Civil Service, Oxford and Cambridge * Local, and other examinations.

W. T. K.

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ALGEBRAIC FACTORS.

PART I.

HOW TO FIND THE FACTORS.

Case I.

WHEN an expression is divisible throughout by any quantity, that quantity is, of course, a factor of the expression, and in practice should be almost always separated from it.

Examples.—1. $mx + my + mz = m(x + y + z)$.

2. $5ax^2 - 10bx^2 + 15cx^2 = 5(a - 2b + 3c)x^2$.

Frequently it happens that one factor is common to some of the terms of an expression, and another to others, and on combining the two sets we can often split up the whole expression.

Examples.—1. $ax + az + by + bz + ay + bx$

$$= (ax + ay + az) + (bx + by + bz)$$
$$= a(x + y + z) + b(x + y + z).$$

Here we have $x + y + z$ in both expressions. Consequently the whole expression $= (a + b)\ (x + y + z)$.

2. $7ab^2 - 5b^3 + 7ac^2 - 5bc^2 - 14abc + 10b^2c$

$$= 7a(b^2 + c^2 - 2bc) - 5b(b^2 + c^2 - 2bc)$$
$$= (7a - 5b)\ (b^2 - 2bc + c^2).$$

EXERCISE I.

Factorize the following:—

1. $10a^2b^2 + 15ab^3 - 25ab^2c$.

2. $55x^3y^4z^4 - 77x^3y^3z^5 + 143x^2y^4z^5$.
3. $30a^2m - 18am^2 - 6a^2m^2$.
4. $5x^6 - 15x^5 + 35x^4 - 75x^3$.
5. $ac + ad + bc + bd$.
6. $2ax - 3by - 2ay + 3bx$.
7. $2bd - 5be + 2cd + 2ad - 5ce - 5ae$.
8. $3m^2x^2 - 5m^2y^2 + 9x^2 - 15y^2$.
9. $6cx^2 - 15cy^2 + 10by^2 - 2ax^2 + 5ay^2 - 4bx^2$.
10. $7b^2m^2 - 3a^2mn - 21b^2mn + a^2m^2 - 10abn^2 + 35b^2n^2 - 2abm^2 + 6abmn + 5a^2n^2$.
11. $c^5x^2 - 5ay^2 - 11b^3z^2 + 11c^5z^2 + 11az^2 + ax^2 + 5b^3y^2 - b^3x^2 - 5c^5y^2$.
12. $24am^2n^2y - 8bm^2n^2x + 8am^2n^2x - 24bm^2n^2y$.

As several of the processes adopted in factorizing depend on the knowledge of certain definite forms, the pupil should now learn the following results, all of which may be verified by multiplication :—

$(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 - 2ab + b^2$,
$(a+b)(a-b) = a^2 - b^2$, $(a-b)(a^2 + ab + b^2) = a^3 - b^3$.
$(a+b)(a^2 - ab + b^2) = a^3 + b^3$.
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Case II.

Since $(a+b)(a-b) = a^2 - b^2$, it is clear that any expression of the form $a^2 - b^2$ can be resolved into two factors of the forms $a+b$ and $a-b$.

Examples.—1. $9x^2 - 16y^2$.

Here $9x^2 = (3x)^2$, and $16y^2 = (4y)^2$; hence if $9x^2$ be represented by a^2, and $16y^2$ by b^2, the equivalent of a is $3x$, and that of b is $4y$.

Thus $9x^2 - 16y^2 = (3x + 4y)(3x - 4y)$.

2. $\frac{25}{x^4} - \frac{x^4}{25}$.

Here $a^2 = \frac{25}{x^4}$, $b^2 = \frac{x^4}{25}$. Thus $a = \frac{5}{x^2}$, $b = \frac{x^2}{5}$, and consequently our factors are $\left(\frac{5}{x^2} + \frac{x^2}{5}\right)\left(\frac{5}{x^2} - \frac{x^2}{5}\right)$.

The process may often be extended, thus:

$$x^{16} - y^{16} = (x^8 - y^8)(x^8 + y^8).$$
$$\text{But } (x^8 - y^8) = (x^4 - y^4)(x^4 + y^4),$$
$$\text{and } (x^4 - y^4) = (x^2 - y^2)(x^2 + y^2).$$
$$\text{Also } (x^2 - y^2) = (x - y)(x + y).$$

Therefore $x^{16} - y^{16} = (x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$.

The reverse operation is simple.

Example.—Multiply $a - b + c$ by $a - b - c$.

Clearly $\{(a - b) + c\}\{(a - b) - c\} = (a - b)^2 - c^2 = a^2 - 2ab + b^2 - c^2$.

Exercise 2.

Resolve into elementary factors—

1. $4a^2 - 9b^2$.
2. $\frac{x^2}{16} - \frac{16}{x^2}$.
3. $36x^2y^2 - 1$.
4. $(x + y)^2 - x^2y^2$.
5. $(m - n^2) - p^2$.
6. $m^2 - (n - p)^2$.
7. $x^4 - 81$.
8. $a^8 - b^8$.
9. $x^2 + y^2 - z^2 + 2xy$.
10. $2ab - a^2 - b^2 + c^2$.
11. $(a + b)^2 - (c - d)^2$.
12. $1 + \frac{y^2 + z^2 - x^2}{2yz}$.

Write down the products of—

13. $(3x - 7)(3x + 7)$.
14. $\left(\frac{a}{2} + \frac{2}{a}\right)\left(\frac{a}{2} - \frac{2}{a}\right)$.
15. $(3abc - 5)(3abc + 5)$.

16. $(a+b+ac)(a+b-ac)$.

17. $(3x-2y)(3x+2y)(9x^2+4y^2)$.

18. $\left(a+\frac{1}{a}+1\right)\left(a-\frac{1}{a}-1\right)$.

19. $(a-b-c-d)(a+b-c+d)$.

20. $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right)$.

21. $\left(a+b-\frac{1}{c}\right)\left(a-b-\frac{1}{c}\right)$.

22. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right)$.

Case III.

Since $(a-b)(a^2+ab+b^2)=a^3-b^3$, it follows that any expression of the form a^3-b^3 can be resolved into factors of the forms $a-b$ and a^2+ab+b^2.

Examples.—1. $(27x^3-125)$.

Here $a^3=27x^3$ and $b^3=125$; hence $a=3x$, $b=5$, $a^2=9x^2$, $ab=15x$, $b^2=25$.

Thus $27x^3-125=(3x-5)(9x^2+15x+25)$.

2. $\left(m^3-\frac{1}{m^3}\right)$.

Here $a^3=m^3$ and $b^3=\frac{1}{m^3}$; hence $a=m$, $b=\frac{1}{m}$, $a^2=m^2$, $ab=1$, $b^2=\frac{1}{m^2}$.

Thus $m^3-\frac{1}{m^3}=\left(m-\frac{1}{m}\right)\left(m^2+1+\frac{1}{m^2}\right)$.

Exercise 3.

Factorize—

1. $8a^3-27b^3$
2. $125p^3-1$.
3. $\frac{64}{27}x^3-\frac{27}{64}y^3$.

4. $(x-y)^3-(x+y)^3$.
5. $(a+b)^3-512$.
6. $c^3-(c-d)^3$.
7. $\frac{x^3}{y^3}-\left(\frac{x+y}{z}\right)^3$.
8. $1-\left(\frac{m+n}{5}\right)^3$.
9. $\frac{a^3}{b^3}-\frac{b^3}{a^3}$.
10. $\frac{a^3}{8}-8$.

Case IV.

Since $(a+b)\ (a^2-ab+b^2)=a^3+b^3$, it follows that any expression of the form a^3+b^3 can be resolved into factors of the forms $a+b$ and a^2-ab+b^2.

Examples.—1. $1000x^6+1$.

Here $a^3=1000x^6, b^3=1$; hence $a=10x^2, b=1, a^2=100x^4$, $ab=10x^2$, $b^2=1$.

Thus $(1000x^6+1)=(10x^2+1)\ (100x^4-10x^2+1)$.

2. $8a^3b^6c^9+x^3$.

Here $a^3=8a^3b^6c^9$, $b^3=x^3$; hence $a=2ab^2c^3$, $b=x$, $a^2=4a^2b^4c^6$, $ab=2ab^2c^3x$, $b^2=x^2$.

Thus $8a^3b^6c^9+x^3=(2ab^2c^3+x)\ (4a^2b^4c^6-2ab^2c^3x+x^2)$.

Exercise 4.

Factorize—

1. $8a^3+27b^3$.
2. $125p^3+1$.
3. $(a-c)^3+(b+c)^3$.
4. $x^3+(x-y)^3$.
5. x^6+y^6.
6. x^9+y^9.
7. $\left(\frac{a+b}{c}\right)^3+\frac{b^3}{c^3}$.
8. $1+\left(\frac{a+b}{3}\right)^3$.
9. $x^3+\frac{1}{x^3}$.
10. $\frac{a^{12}}{8}+\frac{8}{a^{12}}$.

Case V.

Sometimes an expression does not appear to belong to any known form, but by adding and subtracting the same

quantity, it can be thrown into a shape which we at once recognise.

Examples.—1. $x^4 + a^2x^2 + a^4$.

Add and subtract a^2x^2. Then we have

$$(x^4 + 2a^2x^2 + a^4) - a^2x^2 = (x^2 + a^2)^2 - a^2x^2,$$

an expression of the form $a^2 - b^2$, so that we at once write its factors $(x^2 + a^2 + ax)$ $(x^2 + a^2 - ax)$ or $(x^2 + ax + a^2)$ $(x^2 - ax + a^2)$.

2. $a^3 + a^{\frac{3}{2}}b^{\frac{3}{2}} + b^3$

$$= (a^3 + 2a^{\frac{3}{2}}b^{\frac{3}{2}} + b^3) - a^{\frac{3}{2}}b^{\frac{3}{2}}$$
$$= (a^{\frac{3}{2}} + b^{\frac{3}{2}} + a^{\frac{3}{4}}b^{\frac{3}{4}})\ (a^{\frac{3}{2}} + b^{\frac{3}{2}} - a^{\frac{3}{4}}b^{\frac{3}{4}}).$$

Some examples which fall under this case would easily be missed by a casual observer, but a little examination generally suffices to detect the requisite quantity to be added and subtracted.

3. $a^4 - 22a^2b^2 + 49b^4$.

Here we require either the square of $a^2 - 7b^2$ or that of $a^2 + 7b^2$. Now $(a^2 - 7b^2)^2 = a^4 - 14a^2b^2 + 49b^4$, and in order to obtain this we must add to the original expression $8a^2b^2$, which is not a complete square. Next take $(a^2 + 7b^2)^2$, which is $= a^4 + 14a^2b^2 + 49b^4$. To obtain this we must add $36a^2b^2$, which is a complete square.

Thus $a^4 - 22a^2b^2 + 49b^4 = (a^2 + 7b^2)^2 - 36a^2b^2$

$$= (a^2 + 6ab + 7b^2)\ (a^2 - 6ab + 7b^2).$$

Exercise 5.

Convert each of the following into the difference of two squares, and hence factorize them :—

1. $a^4 + 9a^2 + 81$.
2. $16m^4 + 36m^2n^2 + 81n^4$.
3. $(a + b)^4 + (a^2 - b^2)^2 + (a - b)^4$.

4. $x^8 + x^4y^4 + y^8$.
5. $1 + x^2 + x^4$.
6. $(a+3)^4 + 3(a+3)^2 + 4$.
7. $9a^4 + 45a^2b^2 + 81b^4$.
8. $25x^4 - 31x^2y^2 + 9y^4$.
9. $m^4 + 64n^4$.
10. $a^4 + \frac{4}{b^4}$.
11. $49x^4 + 5x^2y^2 + 81y^4$.
12. $121a^4 - 795a^2b^2 + 9b^4$.

Case VI.

Since $a^6 - b^6$ may be regarded either as the difference of two squares or the difference of two cubes, it follows that expressions of the form $a^6 - b^6$ can be factorized in two different ways.

Thus, combining Cases II., III., and IV. :—

$$a^6 - b^6 = (a^3 - b^3)(a^3 + b^3)$$
$$= (a-b)(a^2 + ab + b^2)(a+b)(a^2 - ab + b^2).$$

Or, combining Cases III., II., and VI. :—

$$a^6 - b^6 = (a^2 - b^2)(a^4 + a^2b^2 + b^4)$$
$$= (a+b)(a-b)(a^2 + ab + b^2)(a^2 - ab + b^2).$$

It will be seen that both operations produce the same result, but the former is generally the more convenient.

Example.—$64x^6 - 729y^6$.

First method :—$(8x^3 - 27y^3)(8x^3 + 27y^3)$
$= (2x - 3y)(4x^2 + 6xy + 9y^2)(2x + 3y)(4x^2 - 6xy + 9y^2)$.

Second method :—$(4x^2 - 9y^2)(16x^4 + 36x^2y^2 + 81y^4)$
$= (2x + 3y)(2x - 3y)(4x^2 + 6xy + 9y^2)(4x^2 - 6xy + 9y^2)$.

The process may of course be carried on to any extent until we reach elementary factors, *i.e.* those which cannot be split up any further.

The following example will be instructive:—

$$a^{24}-1=(a^{12}-1)(a^{12}+1)$$
$$=(a^6-1)(a^6+1)(a^{12}+1)$$
$$=(a^3-1)(a^3+1)(a^6+1)(a^{12}+1).$$

Now $a^3-1=(a-1)(a^2+a+1)$,

$$a^3+1=(a+1)(a^2-a+1),$$
$$a^6+1=(a^2+1)(a^4-a^2+1),$$
$$a^{12}+1=(a^4+1)(a^8-a^4+1);$$

$$\therefore\ a^{24}-1=(a-1)(a+1)(a^2+1)(a^2-a+1)(a^2+a+1)$$
$$(a^4+1)(a^4-a^2+1)(a^8-a^4+1).$$

Exercise 6.

Factorize as far as possible—

1. x^6-y^6.
2. $x^{30}-64$.
3. $\frac{1}{64}a^{12}-\frac{1}{15625}$.
4. $a^6b^6-b^6c^6$.
5. $(a-1)^6-(a-2)^6$.
6. $\left(1-\frac{a}{a+b}\right)^6-\left(1+\frac{a}{a+b}\right)^6$.
7. $\overline{5x}|^{12}-\overline{3y}|^{12}$.
8. $\left(\frac{a+b}{b-c}\right)^6-\left(\frac{a-b}{b-c}\right)^6$.

Case VII.

Let us multiply $x+5$ by $x+7$ and carefully examine the process.

$$\begin{array}{l} x+5 \\ x+7 \\ \hline x^2+5x \\ \qquad 7x+35 \\ \hline x^2+12x+35 \end{array}$$

The first term of our product is obtained by multiplying x by x, the second by *adding* the products of x by 5 and x by 7, and the third is the product of 7 and 5.

Now, if we can reverse this process we shall be able to obtain the elementary factors of $x^2+12x+35$. It is clear that the factors of x^2 must be x and x, and those of 35 are 7 and 5; also 7 *added to* 5 make 12, the coefficient of our middle term.

Thus $x^2+12x+35=(x+5)(x+7)$.

The above is a very simple example, but the principles elucidated apply to all expressions of the form x^2+ax+b. It will appear, as we proceed with our subject, that the signs of the various terms are of the utmost importance in directing us to the choice of correct factors, and it will therefore be wise for us to notice what they teach as each case presents itself.

In the above example the *positive* sign of the third term shows that we must *add* the factors of 35 in order to obtain 12, and also that the signs of those factors must be *like*, since $+5$ and $+7$, or -5 and -7 will each give as a product $+35$.

We now turn to the sign of the second term to determine whether these like signs shall be both $+$ or both $-$, and since we find $+12x$ we decide that they must both be $+$.

From the above we conclude:—An expression of the form x^2+ax+b can be split into factors $(x+c)(x+d)$ if $c+d=a$, and $c\times d=b$.

Examples.—$(x^2+8x+12)=(x+2)(x+6)$,

$(x^2+7xy+12y^2)=(x+3y)(x+4y)$.

Exercise 7.

Resolve into elementary factors—

1. $x^2+15x+26$.
2. $a^2+15ab+36b^2$.
3. $x^2+15xy+14y^2$.
4. $x^2+300x+299$.
5. $m^2+50mn+96n^2$.
6. $a^2b^2+17abcd+42c^2d^2$.

7. $(a+b)^2+11(a+b)+18$.
8. $(x+y)^2+23(x+y)\ (y+z)+90(y+z)^2$.
9. $(a+9)^2+10(a+9)+9$.
10. $(x+y)^2+19(x^2-y^2)+60(x-y)^2$.
11. $a^6+19a^3b^3+18b^6$.
12. $x^6+31x^3y^3+204y^6$.

Case VIII.

Multiply $x-3y$ by $x-4y$.

$$\begin{array}{l} x-3y \\ x-4y \\ \hline x^2-3xy \\ \quad -4xy+12y^2 \\ \hline x^2-7xy+12y^2 \end{array}$$

In the examination of this process we discover that it almost exactly resembles Case VII., the exception being the sign of the second term. If then we were to attempt to factorize $x^2-7xy+12y^2$, we should be guided by the same principles as in the last case. The sign of the third term shows that the factors of 12 must be *added* to make 7, and also that they must be of *like* signs. Then the sign of the second term shows that they must both be $-$.

Hence $(x^2-7xy+12y^2)=(x-3y)\ (x-4y)$.

From the above we conclude:—An expression of the form x^2-ax+b can be split up into factors $(x-c)\ (x-d)$ if $c+d=a$, and $c\times d=b$.

Examples.—$(a^2-17a+52)=(a-4)\ (a-13)$,
$(x^2y^2-25xyz+100z^2)=(xy-5z)\ (xy-20z)$.

Exercise 8.

Find the elementary factors of—

1. $x^2-65x+186$.

2. $p^2 - 33pq + 140q^2$.
3. $a^4 - 51a^2b^3 + 144b^6$.
4. $x^8 - 95x^4 + 924$.
5. $m^2n^2 - 73mnpq + 1200p^2q^2$.
6. $a^8b^{10} - 29a^4b^5c^3d^3 + 210c^4d^6$.
7. $(x-5)^2 - 8(x-5) + 12$.
8. $(a-2)^2 - 23(a-2)(a-3) + 76(a-3)^2$.
9. $a^4 - 26a^2b^2 + 153b^4$.
10. $x^8 - 41x^4y^4 + 400y^8$.
11. $m^6 - 14m^3 + 13$.
12. $a^{18} - 35a^9b^9 + 216b^{18}$.

Case IX.

Multiply $x - 3y$ by $x + 7y$, and $x + 3y$ by $x - 7y$.

$$\begin{array}{l} x - 3y \\ x + 7y \\ \hline x^2 - 3xy \\ \quad\;\; 7xy - 21y^2 \\ \hline x^2 + 4xy - 21y^2 \\ \hline\hline \end{array} \qquad \begin{array}{l} x + 3y \\ x - 7y \\ \hline x^2 + 3xy \\ \quad - 7xy - 21y^2 \\ \hline x^2 - 4xy - 21y^2 \\ \hline\hline \end{array}$$

As in the previous examples, the first term in the product is produced by multiplying x by x, and the third term by multiplying $7y$ by $3y$; but the second term is obtained by multiplying x by $7y$ and x by $3y$, and *taking the difference* of the results. It will be found that the signs of the various terms furnish us, as before, with the modes of procedure. The sign of the third term shows that the factors must be *subtracted* to make 4, and also that they must be of *unlike* signs. Now, turning to the second term, we see that *if its sign be* +, *the greater of the factors of* 21 *must be positive, and if* −, *negative.*

From the above we conclude:—An expression of the

We have now obtained the correct coefficients, and it only remains to get the signs right. The third sign of our product shows that they must be like, and the second that they are both $+$. Hence our factors are $(6x+7)(4x+3)$.

2. $24a^2-65ab+21b^2$. By a similar method we obtain $(8a-3b)(3a-7b)$, the only difference in the process being that both signs must be negative on account of the $-65ab$.

3. $24a^2+47a-21$. The third sign shows that we must subtract, and after trial we obtain the numbers $\begin{matrix}8 & & 3\\ & \times & \\ 3 & & 7\end{matrix}$ where $56-9=47$. The third sign also shows that the signs of the second terms must be unlike, and the second sign being $+$ shows that the greater of the cross products must be positive. Hence we must have $+56$ and -9, so that our factors are $(8a-3)(3a+7)$.

4. $24a^2b^2-10abcd-21c^2d^2$. The third sign shows that we must subtract, and after trial we obtain $\begin{matrix}6 & & 7\\ & \times & \\ 4 & & 3\end{matrix}$ where $28-18=10$. The third sign also shows that the signs of our second terms must be unlike, and the second sign being $-$ shows that the greater of the cross products must be negative. Hence we must have -28 and $+18$, so that our factors are $(6ab-7cd)(4ab+3cd)$.

Exercise 10.

What are the factors of—

1. $16x^2+43x+27$.
2. $16x^2-62xy+27y^2$.
3. $16a^2-104ab-27b^2$.
4. $16x^2+431xy-27y^2$.
5. $16m^2-218mn+27n^2$.
6. $16-24x-27x^2$.
7. $16x^2y^2z^2+39xyz-27$.
8. $16a^2+42ab+27b^2$.
9. $16(a+b)^2+147(a+b)(c+d)+27(c+d)^2$.

10. $16(x-4)^2-78(x-4)(y-3)+27(y-3)^2$.
11. $40x^2+61xy-84y^2$.
12. $40a^2-1678abc-84b^2c^2$.
13. $(a+b)^2+(ac+bc)-2c^2$.
14. $abx^2-(ac-bd)x-cd$.
15. $(m-n)x^2-(2m+n)x-3m$.
16. $(m-n)x^2-(2m-3n)x-3m$.
17. $64x^4-148x^2y^2+9y^4$.
18. $8x^6+35x^3y^3+27y^6$.
19. $875x^6+2368x^3y^3-19y^6$.
20. $153x^4-326x^2y^2-275y^4$.
21. $a^{12}-7a^6b^6-8b^{12}$.

Case XI.

Sometimes we can resolve into factors an expression containing three squares with intermediate terms, as, for example:—

$$15a^2-48c^2-8ab-114bc-63b^2-22ac.$$

First arrange the terms thus:—

$$15a^2-8ab-63b^2-22ac-114bc-48c^2.$$

We place a square first, a square third, and a square last. Between a^2 and b^2, ab naturally falls, and the remaining terms must be placed between b^2 and c^2. Now, factorizing the first three terms as in Case x., we obtain $\begin{array}{ccc} 3a & - & 7b \\ 5a & + & 9b \end{array}$. It remains then to find suitable factors for $48c^2$. Suppose we try $4c$ and $12c$ with unlike signs, thus:—

$$\begin{array}{ccccc} 3a & - & 7b & - & 4c. \\ 5a & + & 9b & + & 12c. \end{array}$$

Now $-4c\times+9b=-36bc$, and $+12c\times-7b=-84bc$, which produce $-120bc$, showing that we have the wrong

numbers. Making another attempt with $+8c$ and $-6c$, thus :—

$$\begin{array}{ccccc} 3a & - & 7b & + & 8c, \\ 5a & + & 9b & - & 6c, \end{array}$$

we obtain $+72bc$ and $+42bc = +114bc$, which is the right number, but the wrong sign. Consequently the correct factors to produce $-114bc$ will be $-8c$ and $+6c$. Our result now stands :—

$$\begin{array}{ccccc} 3a & - & 7b & - & 8c. \\ 5a & + & 9b & + & 6c. \end{array}$$

We have lastly to see whether these factors will produce $-22ac$. Multiply across, then, and we get $3a \times 6c = 18ac$, and $5a \times -8c = -40ac$, and these products combined give $-22ac$, as they should do.

Our factors therefore are $(3a - 7b - 8c)\ (5a + 9b + 6c)$.

Sometimes the third square is missing, when the process becomes somewhat simpler.

Example.—$40x^2 - 24xy + 41xz + 9yz - 21z^2$.

Arranging the terms as in last example, this becomes $40x^2 + 41xz - 21z^2 - 24xy + 9yz$.

The factors of the first three terms are $\begin{array}{ccc} 5x & + & 7z \\ 8x & - & 3z \end{array}$, and the third places must contain a factor and zero respectively. The terms not yet used are $-24xy + 9yz$, which should be exactly divisible either by $5x + 7z$ or by $8x - 3z$. The latter expression is contained $-3y$ times in the given terms; hence we place $-3y$ in the position opposite to $8x - 3z$, and 0 in the other position, thus :—

$$\begin{array}{ccccc} 5x & + & 7z & - & 3y. \\ 8x & - & 3z & & 0. \end{array}$$

Now on multiplying across we find that $-3y \times 8x = -24xy$, and $-3y \times -3z = +9yz$.

Our factors therefore are $(5x + 7z - 3y)(8x - 3z)$.

EXERCISE 11.

Factorize—

1. $48x^2 + 8y^2 + 9z^2 + 56xy + 42xz + 27yz$.
2. $72a^2 - 86ab + 15ac - 108c^2 + 24b^2 - 36bc$.
3. $23xz - 20y^2 + 24x^2 - 28xy - 77z^2 + 83yz$.
4. $9z^2 - 30xy + 27x^2 - 36xz + 16yz + 7y^2$.
5. $12a^2 - 132ab - 143ac - 156bc - 144b^2 - 12c^2$.
6. $xy - 12y^2 + 11x^2 - 15z^2 - 4xz + 27yz$.
7. $4m^2(n^2 + 2p^2 - 3np) + 5np(5np + 5mn - 6mp)$.
8. $6a^4 + 6b^4 - 5a^2 + 5b^2 - 13a^2b^2 + 1$.
9. $35a^2 + 27b^2 - 66ab + 14ac - 18bc$.
10. $28b^2c^2 - 44c^4 - 13a^2c^2 - 7a^2b^2 + 6a^4$.
11. $4x^6 + 7x^3z^3 - 9y^6 - 5x^3y^3 + 7y^3z^3$.
12. $27x^2y^2 - 39x^2 + 91 - 128y^2 + 45y^4$.

Expressions of irregular form containing powers of a quantity higher than the square are often very puzzling; but there are one or two cases, which we shall now notice, where the factors can be found with very little difficulty.

Case XII.

If any expression contains a factor of the form $x - y$, the sum of the coefficients of the terms will be 0. By the form $x - y$, we understand a binomial whose terms are of opposite sign, having each the coefficient unity.

In the expression $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$, the coefficients are $1 - 4 + 6 - 4 + 1 = 0$; hence there is a factor of the form $x - y$, which must evidently be $a - b$.

Thus our factors are $(a - b)(a^3 - 3a^2b + 3ab^2 - b^3)$. Proceeding in the same way we should find that $a^3 - 3a^2b +$

$3ab^2 - b^3$ contains a factor $a - b$, but we can be saved that labour by noticing that this is one of the forms we tabulated after Case I., and that it is equal to $(a - b)^3$.

Hence $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 = (a - b)^4$.

Again, in $3m^3 - 10m^2 - 59m + 66$, the coefficients are $3 - 10 - 59 + 66 = 0$; hence there is a factor of the form $x - y$, namely $m - 1$.

Thus $3m^3 - 10m^2 - 59m + 66 = (m - 1)\ (3m^2 - 7m - 66)$ $= (m - 1)\ (m - 6)\ (3m + 11)$ by Case X.

EXERCISE 12.

Find, by use of the principle just explained, the elementary factors of—

1. $7x^3 - 16x^2y + 15xy^2 - 6y^3$.
2. $8a^3 - 11a^2 - 23a + 26$.
3. $13x^3 + 13xy^2 + 5y^3 - 31x^2y$.
4. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.
5. $35m^4 - 22m^3 - 88m^2 + 102m - 27$.
6. $4x^3 - 9xy^2 + 9y^3 - 4x^2y$.

Case XIII.

If any expression contains a factor of the form $x + y$, the sum of the coefficients of the odd powers of x is equal to the sum of the coefficients of the even powers of x (reckoning 0 as even), and the same fact applies to the coefficients of y. As in the last case, a factor of the form $x + y$ means a binomial having unity as the coefficient of each term, but in which the signs of the terms are the same.

In the expression $15x^5 - 47x^4 - 14x^3 + 110x^2 - x - 63$, if there is any factor of the form $x + y$, it must be $x + 1$.

Now, sum of coefficients of odd powers of x

$$= 15 - 14 - 1 = 0,$$

and sum of coefficients of even powers of x

$$= -47 + 110 - 63 = 0.$$

Hence $x + 1$ is a factor, and from the fact that the sum of all the coefficients is 0, we also deduce that $x - 1$ is a factor.

Thus the expression $= (x + 1)\ (x - 1)\ (15x^3 - 47x^2 + x + 63)$.

Apply the same principles to the last factor.

Sum of coefficients of odd powers of $x = 15 + 1 = 16$.

" " even " $x = -47 + 63 = 16$.

Hence $x + 1$ is again a factor, and our expression is

$$(x + 1)\ (x + 1)\ (x - 1)\ (15x^2 - 62x + 63).$$

Factorizing $15x^2 - 62x + 63$ by Case X., we obtain as a final result $(x + 1)\ (x + 1)\ (x - 1)\ (3x - 7)\ (5x - 9)$.

EXERCISE 13.

Resolve into elementary factors—

1. $3x^3 + 8x^2y + 22xy^2 + 17y^3$.
2. $2a^3 - 7a^2b - 14ab^2 - 5b^3$.
3. $4 - 2m - 3m^2 + 3m^3$.
4. $6x^4 - 19x^3y - 21x^2y^2 + 39xy^3 + 35y^4$.
5. $2a^4 - 7a^3 + 18a^2 + 7a - 20$.
6. $3x^5 - 2x^4y - 6x^3y^2 + 4x^2y^3 + 3xy^4 - 2y^5$.

Case XIV.

In expressions containing the third power of a letter, and no higher power, it is clear that if there are any factors, one of them must be of the first dimension.

We can often determine by a few trials whether such factors exist or not. Take, for example, $x^3 - 8x^2 + 17x - 6$. The only factors of 6 are 3 and 2, or 6 and 1. Hence the only possible factors of the first dimension are $x \pm 3$, $x \pm 2$, $x \pm 6$, $x \pm 1$. It can be shown by the last two cases

terror into the heart of the boldest, and we should be very anxious to find a factor before we sought it in this way.

We can avoid the task of actually dividing our expression by the trial factors by recollecting the following important fact. If $x-a$ be a factor of any expression, the whole expression will vanish if a be substituted for x.

In the expression $3x^3-10x^2+13x-10$, put $x=2$. Then we have $24-40+26-10=0$; $\therefore x-2$ is a factor, the other being $3x^2-4x+5$.

If we adopt the same method to find a factor of $4x^3+10x^2y+xy^2+21y^3$, we see that the only factors of 21 are 3 and 7, or 21 and 1; $\therefore$ we must substitute for x one of the values $\pm 3y$, $\pm 7y$, $\pm 21y$, $\pm y$. We soon find that if $x=-3y$, the expression becomes $-108y^3+90y^3-3y^3+21y^3=0$. Hence the factor we are seeking is $x-(-3y)$, *i.e.* $x+3y$.

Thus $4x^3+10x^2y+xy^2+21y^3=(x+3y)(4x^2-2xy+7y^2)$.

Exercise 14.

Factorize—

1. $5x^3-62x^2+86x-99$.
2. $3a^3-4a^2b-175ab^2+392b^3$.
3. $8x^3+30x^2y-53xy^2-15y^3$.
4. $40a^4-71a^3-203a^2+180a-36$.
5. $16x^5-32x^4y-40x^3y^2+80x^2y^3+9xy^4-18y^5$.
6. $x^7-7x^6-3x^5+21x^4+3x^3-21x^2-x+7$.

Case XV.

Let us now consider the forms $x^n \pm y^n$. These are of frequent occurrence and of great importance; the pupil will therefore do well to commit to memory the following results:—

If n be even, $x^n - y^n$ is exactly divisible by both $x+y$ and $x-y$; $x^n + y^n$ is exactly divisible by neither.

If n be odd, $x^n - y^n$ is exactly divisible by $x-y$;
$x^n + y^n$,, ,, $x+y$.

The form of the quotient should be carefully noted. It will be seen that in all the cases the powers of x decrease and those of y increase by unity at each term; also if $x-y$ be the divisor, all the terms in the quotient are positive, but if $x+y$ be the divisor, they are alternately positive and negative. Moreover, the remainder, if any, is always of the form $2y^n$.

Thus:—

$$x^6 - y^6 \div x - y = x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5.$$
$$x^6 - y^6 \div x + y = x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5.$$
$$x^6 + y^6 \div x - y = x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 + \frac{2y^6}{x-y}.$$
$$x^6 + y^6 \div x + y = x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5 + \frac{2y^6}{x+y}.$$
$$x^5 - y^5 \div x - y = x^4 + x^3y + x^2y^2 + xy^3 + y^4.$$
$$x^5 - y^5 \div x + y = x^4 - x^3y + x^2y^2 - xy^3 + y^4 - \frac{2y^5}{x+y}.$$
$$x^5 + y^5 \div x - y = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + \frac{2y^5}{x-y}.$$
$$x^5 + y^5 \div x + y = x^4 - x^3y + x^2y^2 - xy^3 + y^4.$$

Examples.—1. Divide $243a^5 - 32b^5$ by $3a - 2b$.

Here $243a^5 = (3a)^5$, and $32b^5 = (2b)^5$, so that our question is identical with:—Divide $x^n - y^n$ by $x - y$ when n is odd. Thus our quotient is

$$(3a)^4 + (3a)^3\,(2b) + (3a)^2\,(2b)^2 + (3a)\,(2b)^3 + (2b)^4$$
$$= 81a^4 + 54a^3b + 36a^2b^2 + 24ab^3 + 16b^4.$$

2. Write down the quotient of $1 + 625x^4 \div 1 + 5x$.

Here $1 = (1)^4$, and $625x^4 = (5x)^4$, so that our question

is identical with :—Divide $x^n + y^n$ by $x + y$ when n is even.

Our quotient therefore is $1 - 5x + 25x^2 - 125x^3 + \frac{1250x^4}{1+5x}$.

Exercise 15.

Divide :—

1. $x^8 - y^8$ by $x - y$.
2. $81a^8 + 1$ by $3a^2 + 1$.
3. $a^3 - 8b^3$ by $a - 2b$.
4. $x^{10} + y^{10}$ by $x^2 - y^2$.
5. $3125a^5b^5 + 243$ by $5ab + 3$.
6. $a^8 + \frac{1}{a^8}$ by $a - \frac{1}{a}$.
7. $a^8 - \frac{1}{a^8}$ by $a^2 + \frac{1}{a^2}$.
8. $1 - 128a^7$ by $1 + 2a$.

Tell which of the following have factors of the form $x - y$ or $x + y$:—

9. $(a+b)^7 - (a-b)^7$.
10. $(a+b)^7 + (a-b)^7$.
11. $x^{12} + 1$.
12. $a^8b^6c^4 - a^4b^6c^8$.

Case XVI.

The following results are very useful, and should be borne in mind by the student :—

(1) $a^3 + b^3 + c^3 - 3abc$ is divisible by $a + b + c$.
(2) $a^3 - b^3 + c^3 + 3abc$ " $a - b + c$.
(3) $a^3 - b^3 - c^3 - 3abc$ " $a - b - c$.
(4) $a^3 + b^3 - c^3 + 3abc$ " $a + b - c$.

Finding the various quotients without division is rather a tedious process, but when once found they are easily remembered, and we will accordingly give them. They are—

(1) $a^2 + b^2 + c^2 - ab - ac - bc$.
(2) $a^2 + b^2 + c^2 + ab - ac + bc$.
(3) $a^2 + b^2 + c^2 + ab + ac - bc$.
(4) $a^2 + b^2 + c^2 - ab + ac + bc$.

In each quotient the terms $a^2 + b^2 + c^2$ occur, and the others follow a very easily recognised rule.

To find the signs of ab, ac, bc respectively, we must take the term $\pm 3abc$, and notice the sign of the quotient when it is divided by a, b, c, with their proper signs in the divisor. Thus, in the first case, we have $-3abc$ and $+a$, therefore we expect to find $-bc$, and so on. In the second case we have $+3abc$ and $+a$; therefore we expect to find $+bc$, and so on.

As stated above, the process of factorizing these expressions is rather tedious, but as it is occasionally required in examination papers, we will now indicate the method to be pursued.

Example.—Factorize $a^3 - b^3 - c^3 - 3abc$.

$$(a-b-c)^3 = \{(a-b)-c\}^3 = (a-b)^3 - 3(a-b)^2c + 3(a-b)c^2 - c^3$$

$$= a^3 - 3a^2b + 3ab^2 - b^3 - 3a^2c + 6abc - 3b^2c + 3ac^2 - 3bc^2 - c^3$$

$$= (a^3 - b^3 - c^3 - 3abc) - (3a^2b - 3ab^2 - 3abc) - (3a^2c - 3abc - 3ac^2) + (3abc - 3b^2c - 3bc^2)$$

$$= a^3 - b^3 - c^3 - 3abc - (a-b-c)(3ab + 3ac - 3bc).$$

But $(a-b-c)^3 = (a-b-c)(a-b-c)^2$

$$= (a-b-c)(a^2 + b^2 + c^2 - 2ab - 2ac + 2bc).$$

Hence $a^3 - b^3 - c^3 - 3abc - (a-b-c)(3ab + 3ac - 3bc)$

$$= (a-b-c)(a^2 + b^2 + c^2 - 2ab - 2ac + 2bc);$$

i.e., $a^3 - b^3 - c^3 - 3abc = (a-b-c)(a^2 + b^2 + c^2 + ab + ac - bc)$.

Exercise 16.

Find the factors of—

1. $x^3 + 3xyz + y^3 - z^3$.
2. $1 - x^3 + 27y^3 + 9xy$.
3. $a^3 - 8b^3 - 27c^3 - 18abc$.
4. $125a^3 + 27b^3 + 90ab - 8$.
5. $x^3 + 8y^3 + 8z^3 - 12xyz$.
6. $27a^3 - b^3 + 8c^3 + 12abc$.

PART II.

HOW TO USE THE FACTORS.

MULTIPLICATION.

We have already shown that the formula $(a+b)\ (a-b) = (a^2-b^2)$ can often be applied to reduce the working in a multiplication sum. Similar uses can be made of any of the other formulæ.

Examples.—**1.** Multiply $3x+5y$ by $3x+5y$.

Using the formula $(a+b)^2 = a^2+2ab+b^2$, it is clear that $(3x+5y)\ (3x+5y) = 9x^2+30xy+25y^2$.

2. Multiply $4x-5y+3z$ by $4x-5y+3z$.

$$\{(4x-5y)+3z\}^2 = (4x-5y)^2+6z(4x-5y)+9z^2$$
$$= 16x^2-40xy+25y^2+24xz-30yz+9z^2.$$

In the same way we can use the formula

$$(a-b)^2 = a^2-2ab+b^2.$$

3. Multiply $25x^2+35xy+49y^2$ by $5x-7y$.

If $5x$ be represented by a and $7y$ by b, we have to multiply a^2+ab+b^2 by $a-b$; hence our product is $(5x)^3-(7y)^3 = 125x^3-343y^3$.

We can use the formula $(a+b)\ (a^2-ab+b^2) = a^3+b^3$ in a similar manner.

4. Multiply $x^4+x^3+x^2+x+1$ by $x-1$.

Remembering the results tabulated under Case xv., we see at a glance that the product is x^5-1.

5. Multiply together $(a+b)$, $(a+5b)$, $(a-5b)$, $(a-b)$.

Taking the expressions in pairs, we have $(a+b)\ (a-b) = a^2-b^2$, and $(a+5b)\ (a-5b) = a^2-25b^2$. The product of all the factors is therefore

$$(a^2-b^2)\ (a^2-25b^2) = a^4-26a^2b^2+25b^4.$$

By arranging like terms in brackets, we can frequently use our formulæ to great advantage.

6. Multiply $x^2 - mx - nx + m^2 - mn + n^2$ by $x + m + n$.

Arrange thus :—

$$\begin{array}{l}
x^2 - (m+n)x + (m^2 - mn + n^2) \\
x + (m+n) \\
\hline
x^3 - (m+n)x^2 + (m^2 - mn + n^2)x \\
\qquad (m+n)x^2 - (m^2 + 2mn + n^2)x + (m^3 + n^3) \\
\hline
x^3 \qquad\qquad - 3mnx \qquad\qquad + m^3 + n^3 \\
\hline
\end{array}$$

The knowledge of our formulæ enabled us to write down products of two or three terms at a time, thus lessening our labour to a considerable extent.

Exercise 17.

Write down the products of—

1. $2x + 5y - 3z$ and $2x + 5y - 3z$.
2. $3x - 7y - 9z$ and $3x - 7y - 9z$.
3. $36x^2 + 42xy + 49y^2$ and $6x - 7y$.
4. $81a^2 - 99ab + 121b^2$ and $9a + 11b$.
5. $25a^4 - 15a^2 + 9$ and $5a^2 + 3$.
6. $49x^6 + 14x^3y^2 + 4y^4$ and $7x^3 - 2y^2$.
7. $a^3 - a^2 + a - 1$ and $a + 1$.
8. $81a^4 + 54a^3b + 36a^2b^2 + 24ab^3 + 16b^4$ and $3a - 2b$.

Find the continued product of—

9. $(3x - 2y)(2x - 3y)(2x + 3y)(3x + 2y)$.
10. $(2x^2 - 1)(2x^2 + 1)(4x^4 + 2x^2 + 1)(4x^4 - 2x^2 + 1)$.
11. $(a - b)(a + b)^2(a^2 - ab + b^2)$.

Perform the following multiplications :—

12. $(a + b)^2x^2 - 2(a + b)x + 1 \times (a + b)^2x^2 + 2(a + b)x + 1$.
13. $x^2 + 2ax + 2bx + a^2 + 2ab + b^2 \times x + a + b$.
14. $x^2 - 2ax + x + 4a^2 + 2a + 1 \times x + 2a - 1$.

DIVISION.

Every process of factorizing is really a division, one factor being the divisor and the other the quotient. The student has already had considerable practice in this operation, so we shall only notice one or two applications of well-known principles.

If we can factorize both dividend and divisor, it sometimes happens that the factors of the divisor are factors of the factors of the dividend. Thus if we have to divide $x^4 - 13x^2y^2 + 36y^4$ by $x^2 - 5xy + 6y^2$, we find that $x^4 - 13x^2y^2 + 36y^4 = (x^2 - 4y^2)(x^2 - 9y^2)$, and $x^2 - 5xy + 6y^2 = (x - 2y)(x - 3y)$. Now $x - 2y$ is a factor of $x^2 - 4y^2$, and $x - 3y$ is a factor of $x^2 - 9y^2$; hence the required quotient is $(x + 2y)(x + 3y) = x^2 + 5xy + 6y^2$.

By the use of brackets we can often perform the operation of division very neatly.

Example.—Divide $x^3 - 3mnx + m^3 + n^3$ by $x + m + n$.

$$
\begin{array}{l}
x + \\
(m+n)
\end{array}
\Big)
\begin{array}{l}
x^3 \qquad\qquad\qquad - 3mnx + (m^3 + n^3) \\
x^3 + (m+n)x^2 \\
\hline
\quad - (m+n)x^2 - 3mnx \\
\quad - (m+n)x^2 - (m^2 + 2mn + n^2)x \\
\hline
\qquad\qquad\qquad (m^2 - mn + n^2)x + (m^3 + n^3) \\
\qquad\qquad\qquad (m^2 - mn + n^2)x + (m^3 + n^3) \\
\hline
\end{array}
\Big(
\begin{array}{l}
x^2 - (m+n)x \\
+ (m^2 - mn + n^2)
\end{array}
$$

Exercise 18.

Divide—

1. $a^4 - 2a^2b^2 + b^4$ by $a^2 - 2ab + b^2$.
2. $a^{12} - b^{12}$ by $a^4 - b^4$.
3. $9x^4 - 13x^2y^2 + 4y^4$ by $3x^2 - 5xy + 2y^2$.
4. $a^4 + 25a^2b^2 + 625b^4$ by $a^2 + 5ab + 25b^2$.
5. $x^3 - ax^2 - px^2 + qx + apx - aq$ by $x - a$.

6. $a^2 - b^2 - c^2 + d^2 - 2ad + 2bc$ by $a - b + c - d$.
7. $x^4 + y^4 + 2x^2y^2 - 2y^2 - 2x^2 + 1$ by $x^2 + y^2 - 1$.
8. $m^4 - (a + b + c + d)m^3 + (ab + ac + ad + bc + bd + cd)m^2$
$- (abc + abd + acd + bcd)m + abcd$ by $m^2 - (b + d)m + bd$.

GREATEST COMMON MEASURE.

When we recollect that the G. C. M. of two quantities is the product of all the factors common to both quantities, we shall naturally expect our power of finding factors to be of great assistance to us, and we shall find that our expectations will be fully realized.

The methods of applying our knowledge can best be shown by a few examples.

Examples.—1. Find the G. C. M. of $a^3 - 3a^2b + 3ab^2 - b^3$ and $a^2 - 2ab + b^2$. Both these forms are at once recognised; we see they are $(a - b)^3$ and $(a - b)^2$. Thus the G. C. M. is obviously $(a - b)^2$.

EXERCISE 19.

Find by inspection the G. C. M. of—

1. $a^2 - b^2$ and $a^3 - b^3$.
2. $a^3 + b^3$ and $a^4 + a^2b^2 + b^4$.
3. $x^2 + 2x^2y^2 + y^4$ and $x^4 - y^4$.
4. $4x^2 - 12xy + 9y^2$ and $4x^2 - 9y^2$.

If both quantities are quadratic expressions, and if they have a common measure, we can immediately detect it by resolving each quantity into its factors.

2. Find the G. C. M. of $12a^2 - 43ab + 35b^2$ and $8a^2 + 26ab - 45b^2$.

Proceeding as in Case X., we find that

$$12a^2 - 43ab + 35b^2 = (3a - 7b)(4a - 5b),$$

and that

$$8a^2 + 26ab - 45b^2 = (2a + 9b)(4a - 5b).$$

Hence $4a - 5b$, the factor common to both, is the G. C. M. required.

EXERCISE 20.

Find the G. C. M. of—

1. $a^2 + 8ab + 15b^2$ and $a^2 + 9ab + 20b^2$.
2. $1 - x - 56x^2$ and $1 - 15x + 56x^2$.
3. $10x^2 - 19xy - 15y^2$ and $10x^2 + 41xy + 21y^2$.
4. $66a^2 - 11ab - 55b^2$ and $121a^2 - 121b^2$.
5. $8a^2 - 74ab + 105b^2$ and $24a^2 - 62ab + 35b^2$.
6. $72x^2 - 361x + 44$ and $56x^2 + 257x - 33$.

It very seldom happens that we can apply the method of Case XI. to a G. C. M., but when it can be done a great saving of time and labour is effected.

3. Find the G. C. M. of $2a^2 - 21b^2 - 45c^2 + ab + 62bc + ac$ and $3a^2 - 21b^2 - 45c^2 - 2ab + 62bc + 6ac$.

Re-arranging and factorizing, we get

$$2a^2 + ab - 21b^2 + ac + 62bc - 45c^2$$
$$= (a - 3b + 5c)\ (2a + 7b - 9c),$$
$$\text{and } 3a^2 - 2ab - 21b^2 + 6ac + 62bc - 45c^2$$
$$= (a - 3b + 5c)\ (3a + 7b - 9c).$$

Hence the G. C. M. is $a - 3b + 5c$.

EXERCISE 21.

Find the G. C. M. of—

1. $20x^2 - xy + 39yz - 6z^2 - 63y^2 + 2xz$ and
   $30x^2 - 19xy + 120yz - 33z^2 - 63y^2 - 37xz$.
2. $10x^4 + 10y^4 - 8 - 29x^2y^2 - 11x^2 + 38y^2$ and
   $4x^4 + 25y^4 - 8 - 20x^2y^2 - 14x^2 + 35y^2$.
3. $112ab - 84a - 96b + 49a^2 + 64b^2 + 36$ and
   $98ab - 98a - 100b + 49a^2 + 48b^2 + 48$.
4. $3x^2y^2 + 8x^2z^2 + 15y^2z^2 + 2xyz(7y - 5x - 11z)$ and
   $6x^2y^2 - 16x^2z^2 + 21y^2z^2 - (4x - 25y - 10z)xyz$.

Cubic expressions, and those of a higher degree, can sometimes be factorized by the methods already explained, but as a rule, the labour involved is greater than that of finding the G. C. M. by the ordinary method. In cases, however, where the factors of one of the expressions can be easily found, it is always wise to find them, even if we do not see our way clear to obtaining the factors of the other expression.

4. Find the G. C. M. of $48a^2 + 16a - 15$ and $24a^3 - 22a^2 + 17a - 5$.

By Case X., $48a^2 + 16a - 15 = (12a - 5)(4a + 3)$.

Now, since 3 is not a factor of 5, it is evident that $24a^3 - 22a^2 + 17a - 5$ is not divisible by $4a + 3$; hence, if there is a common measure at all, it must be $12a - 5$, and on trial this is found to be the case.

EXERCISE 22.

Find the G. C. M. of—

1. $40x^2 - 41xy - 21y^2$ and $15x^3 - 66x^2y + 73xy^2 - 14y^3$.
2. $2x^2 + 11x - 40$ and $8x^3 - 34x^2 + 41x - 15$.
3. $14a^2 - 71ab + 36b^2$ and
   $8a^4 - 48a^3b + 68a^2b^2 - 69ab^3 + 27b^4$.
4. $45m^2 - 64mn - 77n^2$ and
   $175m^4 + 150m^3n - 972m^2n^2 - 726mn^3 + 605n^4$.

Although, as stated above, it is generally a waste of time to attempt to factorize two cubic expressions whose G. C. M. we have to find, it is advantageous to introduce factors at any stage of the process where it can easily be done.

Example.—Find the G. C. M. of $16x^3 - 24x^2 + 45x - 27$ and $32x^3 - 32x^2 - 14x + 15$.

Instead of trying to find factors, let us commence in the ordinary way, thus :—

$$16x^3 - 24x^2 + 45x - 27 \overline{)\, 32x^3 - 32x^2 - 14x + 15 \,(\, 2}$$
$$\underline{32x^3 - 48x^2 + 90x - 54}$$
$$16x^2 - 104x + 69$$

Now, factorizing the remainder, we get $(4x - 23)(4x - 3)$, and the latter factor is obviously the G. C. M.

EXERCISE 23.

Find the G. C. M. of—

1. $2a^4 + 3a^3 - 3a^2 - 7a - 3$ and $8a^3 - 8a^2 - 4a - 3$.
2. $15x^3 - 57x^2y + 79xy^2 - 45y^3$ and
   $15x^4 - 62x^3y + 83x^2y^2 + 19xy^3 - 99y^4$.
3. $8x^3 + 26x^2 - 52x + 21$ and $16x^3 - 40x^2 + 53x - 24$.
4. $45x^4 - 106x^3y + 117x^2y^2 + 23xy^3 - 35y^4$ and
   $63x^4 - 116x^3y + 117x^2y^2 - 112xy^3 + 40y^4$.

LEAST COMMON MULTIPLE.

If factorial analysis is useful in finding a G. C. M., it is absolutely indispensable in obtaining a L. C. M., since the latter process consists entirely in finding factors by some means or other.

A few examples will at once illustrate and explain our meaning.

Examples.—1. Find the L. C. M. of $24a^3b^2c$, $36a^2b^3c$, and $40ab^3c^2$.

Here $24a^3b^2c = 2^3 \times 3 \times a^3 \times b^2 \times c$,
$36a^2b^3c = 2^2 \times 3^2 \times a^2 \times b^3 \times c$,
$40ab^3c^2 = 2^3 \times 5 \times a \times b^3 \times c^2$.

Now the L. C. M. of the given quantities is the least quantity which contains all these factors. It is

$$2^3 \times 3^2 \times 5 \times a^3 \times b^3 \times c^2 = 360a^3b^3c^2.$$

2. Find the L. C. M. of $8(a^3 - b^3)$, $12(a^2b - ab^2)$, $9(a^2 - b^2)$.

Here $8(a^3 - b^3) = 8(a - b)(a^2 + ab + b^2)$,

$12(a^2b - ab^2) = 12ab(a - b)$,

$9(a^2 - b^2) = 9(a - b)(a + b)$.

Thus L. C. M. $= 72ab(a + b)(a - b)(a^2 + ab + b^2)$

$= 72ab(a + b)(a^3 - b^3)$,

or $72ab(a^2 - b^2)(a^2 + ab + b^2)$.

3. Find the L. C. M. of $8a^2 + 18ab - 35b^2$, $6a^2 + 11ab - 35b^2$, and $12a^2 - 35ab + 25b^2$.

$$8a^2 + 18ab - 35b^2 = (4a - 5b)(2a + 7b),$$
$$6a^2 + 11ab - 35b^2 = (2a + 7b)(3a - 5b),$$
$$12a^2 - 35ab + 25b^2 = (4a - 5b)(3a - 5b).$$

Thus L. C. M. $= (4a - 5b)(2a + 7b)(3a - 5b)$.

4. Find the L. C. M. of $16x^3 - 24x^2 + 45x - 27$, and $32x^3 - 32x^2 - 14x + 15$.

The G. C. M. of the quantities is $4x - 3$.

Hence $16x^3 - 24x^2 + 45x - 27 = (4x - 3)(4x^2 - 3x + 9)$, and $32x^3 - 32x^2 - 14x + 15 = (4x - 3)(8x^2 - 2x - 5)$.

Thus L. C. M. $= (4x - 3)(4x^2 - 3x + 9)(8x^2 - 2x - 5)$.

Exercise 24.

Find the L. C. M. of—

1. $22a^3x^2y$, $33ax^3y^2$, $44bx^2y^3$.
2. $12a^3b$, $16ac^3$, $20ab^2c$, $24a^2bc$, $30abc^2$.
3. $4a^2(x + y)$, $6b^2(x - y)$, $9c^2(x^2 - y^2)$.
4. $30a(a - b)^2$, $15(a^3 - b^3)$, $12a^2(a + b)$.
5. $a^5 - a^2b^3$, $a^3b^2 + b^5$, $a^6 - b^6$.
6. $2x^2 - 18$, $4x^2 + 24x + 36$, $3x^2 - 18x + 27$.
7. $x^2 - 5x + 6$, $x^2 - 6x + 8$, $x^2 - 7x + 12$.
8. $32x^2 - 20xy - 63y^2$, $44x^2 - 89xy + 21y^2$, $16x^2 - 56xy + 49y^2$, $64x^2 + 144xy + 81y^2$.
9. $a^3 - b^3$, $a^3 + b^3$, $a^3 + 2a^2b + 2ab^2 + b^3$, $a^3 - 2a^2b + 2ab^2 - b^3$.

$x^3 - 3x^2y - 9xy^2 + 27y^3$, $3x^3 - x^2y - 27xy^2 + 9y^3$.

FRACTIONS.

Since almost all the operations involved in the simplifications of fractions require the use of the two last rules, it is evident that factors will be continually called into requisition. No new principles will require explaining, since the rules of arithmetic apply to algebraic fractions. We shall therefore confine ourselves to illustrations and examples on the various rules, with only an occasional word of comment.

Reduction to Lowest Terms.

Example.—$\dfrac{3a^3 - 6a^2b + ab^2 - 2b^3}{a^3 - 7ab^2 + 6b^3}$.

The denominator is easily split up into $(a - b)(a - 2b)(a + 3b)$. Now $a - b$ is not a factor of the numerator, since the sum of its coefficients is not 0, and it is clear that $3b$ is not a factor of $2b^3$; hence the G. C. M., if any, must be $a - 2b$. Dividing numerator and denominator by $a - 2b$, we obtain $\dfrac{3a^2 + b^2}{(a - b)(a + 3b)}$.

EXERCISE 25.

Reduce to their lowest terms—

1. $\dfrac{56x^2 - 19x - 15}{56x^2 - 12x - 20}$.
2. $\dfrac{a^3 - 8b^3}{a^3 - 6a^2b + 12ab^2 - 8b^3}$.
3. $\dfrac{12x^2 + 17xy - 7y^2}{15x^3 - 11x^2y + 11xy^2 - 3y^3}$.
4. $\dfrac{x^4 - 21x + 8}{8x^4 - 21x^3 + 1}$.
5. $\dfrac{(a + b)^2 - (c + d)^2}{(a + d)^2 - (b + c)^2}$.
6. $\dfrac{x^2 - y^2 - z^2 + 2yz}{x^2 + y^2 - z^2 + 2xy}$.

Addition and Subtraction.

Simplify $\frac{3}{1-2x} - \frac{7}{1+2x} - \frac{20x-4}{1-4x^2}$.

The L. C. M. of the denominators is $1-4x^2$, thus the result is $\frac{3(1+2x)-7(1-2x)-(20x-4)}{1-4x^2} = \frac{0}{1-4x^2} = 0$.

Add together

$$\frac{1}{a^2+3ab+2b^2}, \frac{1}{a^2+5ab+6b^2}, \text{ and } \frac{1}{a^2+7ab+12b^2}.$$

Factorizing, we obtain—

$$\frac{1}{(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)} + \frac{1}{(a+3b)(a+4b)}$$

$$= \frac{(a+3b)(a+4b)+(a+b)(a+4b)+(a+b)(a+2b)}{(a+b)(a+2b)(a+3b)(a+4b)}$$

$$= \frac{3a^2+15ab+18b^2}{(a+b)(a+2b)(a+3b)(a+4b)}$$

$$= \frac{3(a+2b)(a+3b)}{(a+b)(a+2b)(a+3b)(a+4b)}$$

$$= \frac{3}{(a+b)(a+4b)}.$$

EXERCISE 26.

Simplify—

1. $\frac{1}{x+3y} + \frac{6y}{x^2-9y^2} - \frac{1}{3y-x}$.

2. $\frac{1}{x^2-4x+3} - \frac{1}{x^2-3x+2} - \frac{1}{x^2-5x+6}$.

3. $\frac{20a^2+27a+9}{15a^2+19a+6} + \frac{20a^2+27a+9}{12a^2+17a+6}$.

4. $\frac{2x}{(a-2x)^2} - \frac{a-x}{a^2-5ax+6x^2} + \frac{2}{a-3x}$.

5. $\frac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}} - \frac{a+b}{a^{\frac{1}{3}}+b^{\frac{1}{3}}}$.

6. $\dfrac{m}{n} - \dfrac{(m^2-n^2)x}{n^2} + \dfrac{m(m^2-n^2)x^2}{n^2(n+mx)}$.

7. $\dfrac{a^2}{a^2-ab-ac+bc} + \dfrac{b^2}{b^2-ab-bc+ac} + \dfrac{c^2}{c^2-ac-bc+ab}$.

8. $\dfrac{2x}{\sqrt{3}+1} - \dfrac{2x}{\sqrt{3}-1} + \dfrac{1}{a+\sqrt{a^2-1}} + \dfrac{1}{a-\sqrt{a^2-1}}$.

9. $\dfrac{1}{2a^2-4a+2} + \dfrac{1}{2a^2+4a+2} + \dfrac{1}{1-a^2}$.

10. $\dfrac{x+y-z}{x-y+z} - \dfrac{x-y+z}{x+y-z} - \dfrac{4(y-z)^2}{x^2-(y-z)^2}$.

11. $\dfrac{1}{a(a-b)(a-c)} + \dfrac{1}{b(b-c)(b-a)} + \dfrac{1}{c(c-a)(c-b)}$.

12. $\dfrac{1}{a-2b} - \dfrac{4}{a-b} + \dfrac{6}{a} - \dfrac{4}{a+b} + \dfrac{1}{a+2b}$.

Multiplication and Division.

Simplify $\dfrac{12x^2+x-1}{1-8x+16x^2} \times \dfrac{16x^2-1}{1+6x+9x^2}$.

This becomes $\dfrac{(4x-1)(3x+1)}{(1-4x)(1-4x)} \times \dfrac{(4x-1)(4x+1)}{(1+3x)(1+3x)}$,

or $\dfrac{(4x-1)(3x+1)}{(4x-1)(4x-1)} \times \dfrac{(4x-1)(4x+1)}{(3x+1)(3x+1)} = \dfrac{4x+1}{3x+}$

Simplify $\left(\dfrac{a^2+b^2}{2ab}+1\right)\left(\dfrac{ab^2}{a^3+b^3}\right) \div \dfrac{4a(a+b)}{a^2-ab+b^2}$.

This is equivalent to $\dfrac{(a+b)^2}{2ab} \times \dfrac{ab^2}{a^3+b^3} \times \dfrac{a^2-ab+b^2}{4a(a+b)} = \dfrac{b}{8a}$,

remembering that $a^3+b^3=(a+b)(a^2-ab+b^2)$.

Exercise 27.

Reduce to their simplest forms—

1. $\dfrac{a^3+a^2b+ab^2+b^3}{a^3-a^2b-ab^2+b^3} \times \dfrac{(a+b)^2-3ab}{(a-b)^2+3ab} \times \dfrac{(a-b)^3-(a^3-b^3)}{(a+b)^3-(a^3+b^3)}$.

2. $\left(\frac{a+x}{a^2-ax+x^2}-\frac{a-x}{a^2+ax+x^2}\right)\div\left(\frac{a^2+x^2}{a^3-x^3}-\frac{a^2-x^2}{a^3+x^3}\right).$

3. $\frac{a^4+a^2b^2+b^4}{a^3+b^3}\times\frac{a^2-b^2}{a^3-b^3-2ab(a-b)}.$

4. $\frac{x^4+3x^3+3x^2+x}{y(y^3-1)-3y^3+3y^2}\div\{(x-1)^2+4x\}\times(xy-y)$
$\times(y-1)^2\div(xy+x).$

5. $\frac{a^3b-b^4}{ab^2+a^2b}\div\left\{\frac{a^4+a^3b+a^2b^2}{(a^2-b^2)^3}\div\left(1+\frac{b}{a}\right)^2\right\}.$

6. $\left(\frac{x^2}{y^2}-1\right)\left(\frac{x}{x-y}-1\right)+\left(\frac{x^3}{y^3}+1\right)\left(\frac{x^2+xy}{x^2-xy+y^2}-1\right).$

7. $\frac{20x^2+13x-21}{12x^2+19x-21}\times\frac{12x^2+x-63}{44x^2-47x-117}$
$\times\frac{22x^2+103x+91}{10x^2+49x+49}.$

8. $\frac{6a^2b^2}{x+y}\div\left[\frac{3(x-y)a}{7(c+d)}\div\left\{\frac{4(c-d)}{21ab^2}\div\frac{c^2-d^2}{4(x^2-y^2)}\right\}\right].$

Some more exercises on fractions will be found in the Miscellaneous Examples at the end of the book.

EQUATIONS.

The only use to which factors can be turned in Simple Equations is to assist in reducing fractional equations to their simplest forms. This part of the subject having been already illustrated, we shall now have nothing to say respecting Simple Equations.

In Quadratic Equations, however, the case is altogether different. Here we shall find that the proper use of factors will entirely free us, in the majority of cases, from the disagreeable process known as 'completing the square.'

If an equation can be reduced to the form

$$(ax+b)(cx+d)=0,$$

it can be solved by equating to zero each of the factors

$ax+b$ and $cx+d$. For instance, if $(2x+5)(3x-7)=0$, we first put $2x+5=0$, whence $x=-2\frac{1}{2}$; and then $3x-7=0$, whence $x=2\frac{1}{3}$, thus obtaining both solutions of the quadratic.

Most quadratic equations, when simplified, can be brought to a form capable of being resolved into factors; if we are unable to factorize the expression resulting from the simplification of the various terms, it is a sign that the roots of the equation are surds, in which case we must resort to the usual method of solution.

Examples.—1. Solve $\frac{2x}{3}+\frac{x-1}{15}-\frac{22}{x}=\frac{2}{3}$.

Multiplying by $15x$, we obtain

$$10x^2+x^2-x-330=10x,$$
$$\text{or } 11x^2-11x-330=0,$$
$$i.e.\ x^2-x-30=0.$$

Factorizing, $(x-6)(x+5)=0$.

$$\text{Whence } x-6=0,\ x=6,$$
$$\text{or } x+5=0,\ x=-5.$$

2. $\frac{x^2}{2}-\frac{x}{3}=\frac{1}{32}$.

Multiply by 96. Then $48x^2-32x=3$,

$$\text{or } 48x^2-32x-3=0,$$
$$i.e.\ (12x+1)(4x-3)=0.$$
$$\text{Whence } 12x+1=0,\ x=-\tfrac{1}{12},$$
$$\text{or } 4x-3=0,\ x=\tfrac{3}{4}.$$

EXERCISE 28.

Solve the equations—

1. $x^2-3x=3(9+x)$.
2. $9x^2-7x=2$.
3. $x-\frac{x^3-8}{x^2+5}=2$.

4. $\frac{1}{x-3}+\frac{1}{x-5}=\frac{4}{3}$.

5. $\frac{2x-1}{x+1}+\frac{3x-1}{x+2}=4+\frac{x-7}{x-1}$.

6. $(x-a)(x-b)=a(2a-b)$.

7. $\frac{x^2+b^2}{x^2-b^2}+\frac{2x+3b-a}{x+b}=\frac{2(x^2+bx-b^2)}{x^2-b^2}$.

8. $3x=10-\sqrt{5x+6}$.

9. $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$.

10. $x-\frac{a^2-b^2}{x}=\frac{b^2}{a}$.

Some equations, though not quadratics, can be solved in the same manner, as the following example will show :—

$$2x^2-\sqrt{x^2-2x-3}=4x+9.$$

Re-arranging the terms, we obtain

$$2(x^2-2x-3)-\sqrt{x^2-2x-3}-3=0.$$

Here, if $\sqrt{x^2-2x-3}=a$, the equation is

$$2a^2-a-3=0.$$

Factorizing, $(2a-3)(a+1)=0$.

Whence $a=\frac{3}{2}$ or -1.

Taking $\sqrt{x^2-2x-3}=-1$, we obtain

$$x^2-2x-3=1,$$
$$x^2-2x+1=5,$$
$$i.e., (x-1)^2=5,\ x-1=\pm\sqrt{5}.$$

Whence $x=1\pm\sqrt{5}$.

Taking $\sqrt{x^2-2x-3}=\frac{3}{2}$, we obtain

$$x^2-2x-3=\tfrac{9}{4},$$
$$\text{or } 4x^2-8x-21=0,$$
$$i.e., (2x-7)(2x+3)=0.$$

Whence $x={}^{7}$ or $-{}^{3}$.

EXERCISE 29.

Solve—

1. $\frac{x^2}{6}+\frac{27}{2}=\frac{1}{3}\sqrt{13(x^2-30x+81)}+5x.$

2. $x-5-4\sqrt{x-5}=0.$

3. $3x^2+15x+4=2\{1+\sqrt{x^2+5x+1}\}.$

4. $2x^2+6-3(x+1)-2\sqrt{2x^2-3x+2}=0.$

5. $2x^2-6+14\sqrt{2+11x-2x^2}=11x+36.$

6. $x^5-2x^2=\frac{31}{x}-16x^{-1}.$

7. $x^2-ax=\{\sqrt{x^2-ax+b^2}-b\}b.$

8. $(c^2-x^2)+c\sqrt{c^2-x^2}=12c^2.$

If an equation of a higher degree than the second have any real roots, the best and surest mode of solution is to factorize, and equate each factor to zero, as in the case of a quadratic.

Examples.—1. $x^4-4x^3+8x-165=0.$

We soon find that $x-5$ is a factor; hence one of our answers is $x=5$. Dividing by $x-5$, we get

$$x^3+x^2+5x+33=0.$$

Next we find that another factor is $x+3$; hence another answer is $x=-3$.

Dividing by $x+3$, $x^2-2x+11=0$,

$$\text{or } x^2-2x+1=-10;$$

$$\therefore\ x-1=\pm\sqrt{-10},$$

$$x=1\pm\sqrt{-10}.$$

Thus our results are $5,\ -3,\ 1\pm\sqrt{-10}.$

2. $x(x+2)(x-2)(x-4)=105.$

Multiplying out, and re-arranging, we get

$$x^4-4x^3-4x^2+16x-105=0.$$

Factorizing, $(x-5)(x+3)(x^2-2x+7)=0.$ Whence

$x-5=0$, $x+3=0$, $x^2-2x+7=0$, from which we obtain $x=5$, or -3, or $1\pm\sqrt{-6}$.

EXERCISE 30.

Solve the equations—

1. $(x+3)^2-2(x^2+3)=2x(x+1)^2$.
2. $2x^2-x-2x\sqrt{1-x^2}=1\frac{1}{2}$.
3. $\left(\frac{a^2+ax+x^2}{a^2-ax+x^2}\right)^2=\frac{a+2x}{a-2x}$.
4. $3x\{(x-1)^2+3\}=5x^2+4$.
5. $7x^4-5x^2-2=9x(x^2-1)$.
6. $3(x-4)^3+19(x-1)^2=80(x-2)-1$.

A great number of artifices used in the solution of simultaneous quadratics of two unknowns depend upon the formulæ tabulated after Case 1. We will now give a few of the most important :—

(1)
$$x+y=7, \quad . \quad . \quad . \quad (1)$$
$$x^2+y^2=29. \quad . \quad . \quad . \quad (2)$$

Squaring (1),

$$x^2+2xy+y^2=49. \quad . \quad . \quad (3)$$

Subtracting (2) from (3),

$$2xy=20. \quad . \quad . \quad (4)$$

Subtracting (4) from (2),

$$x^2-2xy+y^2=9.$$

Whence $x-y=\pm 3$.

Combining this result with (1), we obtain

$$x=5 \text{ or } 2,\ y=2 \text{ or } 5.$$

A similar method is applicable to equations of the form

$$x-y=3,$$
$$x^2+y^2=49.$$

(2)
$$x+y=11, \quad . \quad . \quad . \quad (1)$$
$$xy=28. \quad . \quad . \quad . \quad (2)$$

Squaring (1),

$$x^2+2xy+y^2=121.$$

From (2),

$$4xy=112.$$

By subtraction,

$$x^2-2xy+y^2=9,$$

$$x-y=\pm 3.$$

Combining this result with (1), we obtain

$$x=7 \text{ or } 4,\ y=4 \text{ or } 7.$$

A similar method is applicable to equations of the form

$$x-y=3,$$

$$xy=28.$$

(3) $$x^2+y^2=82, \quad . \quad . \quad . \quad (1)$$

$$xy=\ \ 9. \quad . \quad . \quad . \quad (2)$$

From (2),

$$2xy=18.$$

Adding this to (1),

$$x^2+2xy+y^2=100.$$

Subtracting it from (1),

$$x^2-2xy+y^2=64.$$

Hence $x+y=\pm 10$,

$$x-y=\pm 8.$$

Thus $x=\pm 9$ or ± 1, $y=\pm 1$ or ± 9.

(4) $$x^3+y^3=189, \quad . \quad . \quad . \quad (1)$$

$$x+y=\ \ 9. \quad . \quad . \quad . \quad (2)$$

Cubing (2),

$$(x^3+y^3)+3xy(x+y)=729,$$

i.e. $189+27xy=729$.

Whence $xy=20$.

Combining this with (2), as in Example 2, we obtain

$$x=5 \text{ or } 4,\ y=4 \text{ or } 5.$$

A similar method is applicable to equations of the form

$$x^3-y^3=61,$$

$$x-y=1.$$

(5) $$x - y = 2, \quad . \quad . \quad . \quad (1)$$
$$x^4 - y^4 = 544. \quad . \quad . \quad . \quad (2)$$

Assume $x = u + v$,
$y = u - v$.

Then $x - y = 2v = 2$; $\therefore v = 1$.

Also $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$.

Now $x^2 + y^2 = (u + v)^2 + (u - v)^2$
$= (u + 1)^2 + (u - 1)^2 = 2u^2 + 2$.

And $x^2 - y^2 = (u + v)^2 - (u - v)^2$
$= 4uv = 4u$;

$\therefore x^4 - y^4 = 4u(2u^2 + 2) = 544$.

Hence $u^3 + u = 68$.

Multiplying by u,
$$u^4 + u^2 = 68u = 4 \times 17u,$$
$$u^4 + 17u^2 + (\tfrac{17}{2})^2 = 16u^2 + 68u + (\tfrac{17}{2})^2.$$

Taking the square root,
$$u^2 + \tfrac{17}{2} = 4u + \tfrac{17}{2},$$
$$u^2 = 4u, \; u = 4.$$

Thus $x = u + v = 5$, $y = u - v = 3$.

(6) $$3x^2 - 4xy + y^2 = 20, \quad . \quad . \quad (1)$$
$$2x^2 - 7y^2 = 4. \quad . \quad . \quad (2)$$

Dividing (1) by (2), we obtain
$$\frac{3x^2 - 4xy + y^2}{2x^2 - 7y^2} = 5.$$

Whence $3x^2 - 4xy + y^2 = 10x^2 - 35y^2$,
or $7x^2 + 4xy - 36y^2 = 0$,
i.e., $(7x + 18y)(x - 2y) = 0$.
Thus $y = -\frac{7}{18}x$ or $\frac{1}{2}x$.

Taking $y = -\frac{7}{18}x$, and substituting in (2), we get
$$x = \pm \frac{36}{\sqrt{305}}, \; y = \mp \frac{14}{\sqrt{305}}.$$

Taking $y = \frac{1}{2}x$, and substituting in (2), we obtain
$$x = \pm 4, \; y = \pm 2.$$

The above are only a few of the plans adopted in the solutions of simultaneous equations. An acquaintance with all the methods is only to be obtained by practice, and an account of only a small portion of them would occupy more space than the whole of this book. Suffice it to say, that the farther the student penetrates into the subject, the more convinced will he be of the incalculable advantage derived from the ability to handle factors with dexterity. As a large number of equations will be found amongst the Miscellaneous Examples, we shall not give any here, but shall at once proceed to the next subject which claims our attention.

EVOLUTION.

When an expression consists of more than three terms, the best method of extracting its square root is undoubtedly the ordinary one; but in the case of a trinomial, the square root can be at once written down by the use of the formulæ:

$$a^2 + 2ab + b^2 = (a + b)^2,$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

Take, for example, $9x^2 + 24xy + 16y^2$.

Here, if $a^2 = 9x^2$, and $b^2 = 16y^2$, we have $a = 3x$, $b = 4y$, and clearly $2ab = 24xy$, which is our middle term. Hence the square root required is $3x + 4y$, or $3x - 4y$, and the sign of the middle term shows that the former is the correct result.

Again, take $25x^6 - 110x^3y^2 + 121y^4$.

Here $a = 5x^3$, $b = 11y^2$, and on examining the middle term, we find that it is twice the product of these quantities, *i.e.* $2ab$.

Hence the square root required is $5x^3 - 11y^2$, the negative sign corresponding with $-110x^3y^2$.

EXERCISE 31.

Write down the square roots of—

1. $x^2 - 4xy + 4y^2$.
2. $49x^2 + 112xy + 64y^2$.
3. $25a^2 - 170ab + 289b^2$.
4. $9a^4 - 42a^2 + 49$.
5. $(x-2)^2 + 6(x^2 - 5x + 6) + 9(x-3)^2$.
6. $(x^2 - 5)^2 + 14x(x^2 - 5) + 49x^2$.

By the use of the corresponding formulæ for the cube of a binomial, we can extract the cube root of an expression when such root contains only two terms. These formulæ are:

$$a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3,$$
$$a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3.$$

It should be noticed that in the first formula the signs are all positive, while in the second they are alternately positive and negative.

Examples.—1. Find the cube root of

$$8x^3 + 12x^2y + 6xy^2 + y^3.$$

Here $a^3 = 8x^3$, $b^3 = y^3$, so that $a = 2x$ and $b = y$. We have now to examine the intervening terms for the purpose of discovering whether they are of the forms $3a^2b$ and $3ab^2$.

Now $12x^2y = 3 \times 4x^2 \times y = 3a^2b$,
and $6xy^2 = 3 \times 2x \times y^2 = 3ab^2$.

Hence our cube root is evidently $2x + y$.

2. $125x^6 - 225x^4 + 135x^2 - 27$.

Here $a^3 = 125x^6$, $b^3 = 27$, so that $a = 5x^2$ and $b = 3$.

Also $225x^4 = 3 \times 25x^4 \times 3 = 3a^2b$,
and $135x^2 = 3 \times 5x^2 \times 9 = 3ab^2$.

Thus our cube root is $5x^2 - 3$.

EXERCISE 32.

Write down the cube roots of—

1. $x^3+6x^2y+12xy^2+8y^3$.
2. $a^6-3a^4b^2+3a^2b^4-b^6$.
3. $1-21x+147x^2-343x^3$.
4. $8a^6-132a^4b^3+726a^2b^6-1331b^9$.
5. $(a+b)^3-3(a+b)^2(a-b)+3(a+b)(a-b)^2-(a-b)^3$.
6. $8(a+2)^3+12(a+2)^2+6(a+2)+1$.

RATIONALIZING DENOMINATORS.

If the denominator of a fraction contains a surd, and we require to simplify the fraction, we can avoid the inconvenient process of dividing by a long decimal, by a simple operation known as Rationalizing the Denominator. We shall only deal with those containing quadratic surds, and it will be found that the simplification of these depends entirely upon the formula $(a+b)(a-b)=a^2-b^2$.

Suppose the denominator of a fraction is of the form $x+\sqrt{y}$. It is clear that if we multiply both numerator and denominator by $x-\sqrt{y}$, our denominator becomes rational, since $(x+\sqrt{y})(x-\sqrt{y})=x^2-y$. Similarly, if the denominator were $x-\sqrt{y}$, we should multiply by $x+\sqrt{y}$.

Again, if the denominator be of the form $\sqrt{x}\pm\sqrt{y}$, we must multiply by $\sqrt{x}\mp\sqrt{y}$, producing in either case $x-y$.

Also, if the denominator consist of three quadratic surds, it can be rationalized in two operations. Take, for instance, $\sqrt{x}+\sqrt{y}+\sqrt{z}$ as a denominator.

If we multiply both numerator and denominator by $\sqrt{x}+\sqrt{y}-\sqrt{z}$, our denominator will be $x+y-z+2\sqrt{xy}$.

Next, multiplying by $x+y-z-2\sqrt{xy}$, we shall get $(x+y-z)^2-4xy$, a rational quantity.

Examples.—1. Rationalize the denominator of $\frac{4+2\sqrt{3}}{2+\sqrt{2}}$.

Multiplying numerator and denominator by $2-\sqrt{2}$, we get $\frac{(4+2\sqrt{3})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}$, which becomes

$$\frac{8+4\sqrt{3}-4\sqrt{2}-2\sqrt{6}}{4-2}=4+2\sqrt{3}-2\sqrt{2}-\sqrt{6}.$$

2. Find the value of $\frac{5+\sqrt{2}}{5-\sqrt{2}}$ correct to three places of decimals.

The expression $=\frac{(5+\sqrt{2})(5+\sqrt{2})}{(5+\sqrt{2})(5-\sqrt{2})}=\frac{27+10\sqrt{2}}{23}$

$$=\frac{41\cdot1421}{23}=1\cdot7887=1\cdot789 \text{ nearly.}$$

3. Simplify $\frac{\sqrt{2}+\sqrt{5}-\sqrt{3}}{\sqrt{2}+\sqrt{5}+\sqrt{3}}$.

Multiply numerator and denominator by $\sqrt{2}+\sqrt{5}-\sqrt{3}$.

Then $\frac{(\sqrt{2}+\sqrt{5}-\sqrt{3})(\sqrt{2}+\sqrt{5}-\sqrt{3})}{(\sqrt{2}+\sqrt{5}+\sqrt{3})(\sqrt{2}+\sqrt{5}-\sqrt{3})}$

$$=\frac{10+2\sqrt{10}-2\sqrt{6}-2\sqrt{15}}{4+2\sqrt{10}}=\frac{5+\sqrt{10}-\sqrt{6}-\sqrt{15}}{2+\sqrt{10}}.$$

Now, multiplying numerator and denominator by $2-\sqrt{10}$, we obtain

$$\frac{3\sqrt{6}-3\sqrt{10}}{4-10}=\tfrac{1}{2}(\sqrt{10}-\sqrt{6}).$$

EXERCISE 33.

Find the value of the following quantities, each to three places of decimals :—

1. $\dfrac{3}{\sqrt{8}-1}$.

2. $\dfrac{4}{\sqrt{5}-1}$.

3. $\dfrac{\sqrt{7}+2}{\sqrt{7}-2}$.

4. $\dfrac{1}{1-\sqrt{.9}}$.

5. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}+\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.

6. $\dfrac{7-\sqrt{11}}{7+2\sqrt{11}}$.

7. $\dfrac{1}{\sqrt{2}+\sqrt{7}-\sqrt{5}}$.

8. $\dfrac{2+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$.

EVOLUTION OF SURDS.

Bearing in mind the fact (proved in most treatises on algebra) that if

$$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}, \text{ then } \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y},$$

we can easily extract the square root of a binomial surd, by a method which we will now exemplify.

Example.—Find the square root of $8+2\sqrt{15}$.

Assume

$$\sqrt{8+2\sqrt{15}}=\sqrt{x}+\sqrt{y}. \qquad (1)$$

Then

$$\sqrt{8-2\sqrt{15}}=\sqrt{x}-\sqrt{y}. \qquad (2)$$

Hence, by multiplication,

$$\sqrt{64-60}=x-y,$$

$$i.e., \; x-y=2.$$

Now, squaring (1),

$$8+2\sqrt{15}=x+y+2\sqrt{xy},$$

and, since the rational parts on each side are equal, we have $x+y=8$.

Combining this with $x - y = 2$, we obtain $x = 5$, $y = 3$.

Hence $\sqrt{8 + 2\sqrt{15}} = \sqrt{5} + \sqrt{3}$.

It is occasionally possible to extract the square root of an expression of the form $a + \sqrt{b} + \sqrt{c}$ by a somewhat similar process, but as the problem is of very little practical importance, we shall not here insert it. The student who is curious on the subject, and also on the method of extracting the cube root of a binomial surd, may consult Todhunter's *Algebra for the Use of Colleges and Schools.*

EXERCISE 34.

Extract the square root of—

1. $32 + 10\sqrt{7}$.
2. $11 - 6\sqrt{2}$.
3. $12 + 2\sqrt{35}$.
4. $10 - 2\sqrt{21}$.
5. $30 - 12\sqrt{6}$.
6. $100 + 2\sqrt{99}$.

MISCELLANEOUS QUESTIONS.

1. Reduce to its lowest terms $\dfrac{3x^2 + 12x + 9}{x^5 + 5x^3 + 6}$.

2. Give the value of $\dfrac{2 + \sqrt{5} + \sqrt{20}}{1 + \sqrt{45}}$ as far as three decimal places.

3. Solve the equations—

(*a*) $x^3 + y^3 = 65$, $x + y = 5$.

(*b*) $x + \sqrt{x^2 - y^2} = 8$, $x - y = 1$.

(*c*) $\dfrac{25x^2 - 16}{10x + 8} = 3 \cdot \dfrac{x^2 - 4}{2x - 4}$

4. What is the value of $\dfrac{2\sqrt{18} + \sqrt{32}}{\sqrt{56} + 3\sqrt{14}}$ to three decimal places?

5. Divide $(x^3 - 6x^2 + 11x - 6)(x - 4)$ by $x - 1$, and $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

6. Simplify $\frac{x^6-64}{x-2}$ and $(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})(x^{\frac{1}{3}}+y^{\frac{1}{3}})$.

7. Multiply $a^2-2ab+b^2$ by $a^2+2ab+b^2$, and $a-b-c-d$ by $a-b+c+d$.

8. Reduce to its simplest form $\dfrac{\dfrac{\sqrt{x+y}}{\sqrt{x-y}}-\dfrac{\sqrt{x-y}}{\sqrt{x+y}}}{\dfrac{\sqrt{x+y}}{\sqrt{x-y}}+\dfrac{\sqrt{x-y}}{\sqrt{x+y}}}$.

9. Show that $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$ $=(bz-cy)^2+(cx-az)^2+(ay-bx)^2$.

10. Calculate the quotient of the former by the latter of the two expressions—

$$\frac{\dfrac{x^2-a^2}{(x-1)(x-b)}}{\dfrac{(a+b)(x+b)}{(1-x^2)(a-b)}} \text{ and } \frac{\dfrac{a^2-b^2}{(x-a)(x+1)}}{\dfrac{(a+b)(x+b)}{x+a}}.$$

11. Find the product of the four factors—

$$(x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

12. Show that the algebraical expression

$$(cy-bz)^2+(az-cx)^2+(bx-ay)^2+(ax+by+cz)^2$$

is divisible by $x^2+y^2+z^2$.

13. Show that the algebraical expression $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-xz)(z^2-xy)$ is a perfect square.

14. Find the factors of—

(*a*) $1+\dfrac{b^2+c^2-a^2}{2bc}$.

(*b*) $35x^2-61xy+24y^2$.

(*c*) a^6-8b^6.

(*d*) $8x^3-2x^2-11x+15$.

15. Find two numbers whose sum is 13, and the difference of whose squares is 39.

16. Find the numerical value of $\dfrac{a^2+\sqrt{b^2-a^2}}{a^2-\sqrt{b^2-a^2}}+\dfrac{c\sqrt{c^2+1}}{b\sqrt{a^2+9}}$ when $a=5$, $b=6$, $c=4$, to two places of decimals.

17. Find the L. C. M. of $4k^2-9y^2$, $4k^2-y^2$, $4k^2-8ky+3y^2$, $4k^2+4ky-3y^2$, $4k^2-4ky-3y^2$, $4k^2+8ky+3y^2$.

18. Show that if $x=\left(\dfrac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$, then

$$\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\left(\sqrt[p]{x}+\sqrt[q]{x}\right)=\left(\frac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}.$$

19. Find the numerical value of $\dfrac{a(b-\sqrt{c})^2}{\sqrt{b^2-a^2}}$ when $a=3$, $b=4$, $c=5$.

20. Solve the equations—

(*a*) $\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}$.

(*b*) $\dfrac{2x-x^2}{(1-x)^2}=a$.

(*c*) $\sqrt{x+1}+\sqrt{x-1}=\dfrac{2}{\sqrt{x+1}}$.

21. Reduce $\dfrac{\sqrt{1\cdot75}-\sqrt{\cdot63}}{\sqrt{3\cdot5}-\sqrt{2\cdot1}}$ to its simplest form.

22. Find the L. C. M. of $7(a^2-x^2)$, $9(a+x)^2$, $15(a-x)$, $3(a^2+x^2)$, $21(a^4-x^4)$.

23. Multiply $(a+b)^2+(a-b)^2$ by $(a+b)(a-b)$, and divide the result by $a^3+a^2b+ab^2+b^3$.

24. Reduce to a single fraction—

$$\frac{a+x}{a-x}+\frac{a-x}{a+x}+\frac{2ax}{a^2-2ax+x^2}+\frac{a^2+x^2}{a^2+2ax+x^2}.$$

25. Reduce to its simplest form the fraction—

$$\frac{24x^3-82x^2+89x-30}{72x^4-306x^3+469x^2-306x+72}.$$

26. Show that

$$\frac{(m-a)(n-a)}{(a-b)(a-c)}+\frac{(m-b)(n-b)}{(b-c)(b-a)}+\frac{(m-c)(n-c)}{(c-a)(c-b)}=1.$$

27. Reduce to its simplest terms $\frac{36x^4 - 13x^2 + 1}{30x^3 - 19x^2 + 1}$.

28. Of two square fields, one exceeds the other by 100 acres, and its side is longer than that of the other by 400 yards. Find the length of the side of each field.

29. If $ax + by = 1$, and $ax^2 + by^2 = \frac{1}{a+b}$, then $ax^n + by^n = (a+b)^{1-n}$.

30. Find the L. C. M. of $1 - 8x + 17x^2 + 2x^3 - 24x^4$ and $1 - 2x - 13x^2 + 38x^3 - 24x^4$.

31. Solve the equations—

(*a*) $(x+a)(x+3a)(x+6a) = (x+2a)(x+4a)^2$.

(*b*) $x + \sqrt{xy} = 18$, $x - y = 9$.

32. If $\frac{1}{a} + \frac{1}{b} + \frac{1}{a+x} = 0$, and $\frac{1}{a} + \frac{1}{c} + \frac{1}{a+y} = 0$, also $\frac{1}{a} + \frac{1}{x} + \frac{1}{y} = 0$, then $a + b + c = 0$.

33. Find the L. C. M. of $24ax^2y^3$, $45a^2x^2y^2$, $75a^2x^3y$, and $100a^3xy^2$.

34. Reduce to its lowest terms—

$$\frac{x^4 + 4x^3 - 19x^2 - 46x + 120}{x^4 - 25x^2 + 144}.$$

35. Find the G. C. M. of $45x^3y + 3x^2y^2 - 9xy^3 + 6y^4$, and $54x^2y - 24y^3$.

36. Resolve into elementary factors—

$$x^2 + xy + y^2,\ x^2 - y^2 - z^2 + 2yz,$$
$$(x^2 + xy - y^2)^2 - (x^2 - xy - y^2)^2.$$

37. Find the value of $x^3 - 8y^3 + 29z^3 + 18xyz$, when $2y = x + 3z$, and $z = 5$.

38. Divide $(x^6 - y^6)(x^8 - y^8)$ by $(x^3 - y^3)(x^4 - y^4)$. What is the value of the quotient when $x = y = 1$?

39. Solve the equations—

(*a*) $\frac{1+x+x^2}{1-x+x^2}=\frac{62}{63}\cdot\frac{1+x}{1-x}$.

(*b*) $x^2-xy=6,\ x^2+y^2=61$.

(*c*) $\sqrt{x^2+1}-\sqrt{x^2-1}=\frac{x}{\sqrt{x^4-1}}$.

40. Show that $x^8+y^8+(x+y)^8$ is divisible by x^2+xy+y^2 without remainder.

41. If $x=\frac{a+b}{c-d}$, then $(a-cx)^2+(x^2-1)\ (b^2-d^2)$ is a complete square.

42. Solve the equations—

(*a*) $\frac{15}{x-1}-\frac{8}{x-2}=\frac{3}{x-3}$.

(*b*) $x(y+z)=28,\ y(x+z)=18,\ z(x+y)=30$.

(*c*) $x^2+y=11x,\ y^2+x=11y$.

43. Find the square root of $39+\sqrt{1496}$.

44. Factorize $m^2-n^2-p^2+q^2-2(mq-np)$, x^8+x^4+1, $a^{3n+1}b^{-1}-a^{n+1}b^{-3}$, $x^3-(y-z)^3$.

45. There are two numerical quantities such that twice the first added to the reciprocal of the second is equal to the second; while the square of the second, together with the product of both, is equal to 4. Find them.

46. Supposing y to be very much greater than x, find the greatest integral fourth root which is contained in

$$16x^4+32x^3y+24x^2y^2+4xy^3+y^4.$$

47. Solve the equations—

$$x^2+2yz=y^2+2xz=z^2+2xy=3a^2.$$

48. Show that if $p^2-y^2=(x-a)^2$, $q^2-y^2=(x-b)^2$, $r^2-y^2=(x-c)^2$, and a, b, c, are all different, then

$$p^2(b-c)+q^2(c-a)+r^2(a-b)+(b-c)\ (c-a)\ (a-b)=0.$$

49. Divide unity into two parts such that the product of the whole and one of the parts shall be equal to the

square of the other part. Give the answer to three places of decimals.

50. Given $ab - \frac{1}{2}(a+b)(p+q) + pq = 0$,

and $cd - \frac{1}{2}(c+d)(p+q) + pq = 0$.

Show that if a or b is equal to c or d, then p is equal to q, unless $a+b=c+d$.

51. Find the value of $\frac{x}{y}$, having given

$$\frac{x^{2n} - ay^{2n}}{x^{2n} + ay^{2n}} = \frac{x^n - b(x-y)^n}{x^n + b(x-y)^n}.$$

52. Show that

$$2(a-b)(a-c) + 2(b-c)(b-a) + 2(c-b)(c-a)$$

is the sum of three squares.

53. Simplify—

(*a*) $\dfrac{x^3 - \frac{3x}{4} + \frac{1}{4}}{(2x-1)^2}$.

(*b*) $\dfrac{1}{x^2 - 5x + 6} - \dfrac{3}{x^2 - x - 6} + \dfrac{2}{x^2 + x - 6}$.

54. Solve the equations—

(*a*) $ax^2 - bx = a + b$.

(*b*) $x^2 + y = 3$, $x + y^2 = 3$.

55. A man buys a certain number of acres of land for £1600; by selling it at £28 an acre, he gains as much as he gave for 3 acres. How many acres did he buy?

56. Find the factors of $1 - \left(\dfrac{y^2 + z^2 - a^2}{2yz}\right)^2$, $a^4 - 10a^2 + 9$, $2x^2 - x - 1$, $abx^2 + c(a-b)x - c^2$.

57. Given that $a : b :: b : c$, show that

$$a^2 + b^2 + c^2 = (a+b+c)(a-b+c).$$

58. Show that the value of $(a^2 - bc) + (b^2 - ca) + (c^2 - ab)$ is not altered if a, b, and c are each of them increased or diminished by the same quantity.

59. Solve $3x^2-7xy+2y^2=0$, $3x+4y=12$.

60. A carpet, in shape a rectangle, has an area of 315 square feet; if a piece is taken off, so as to reduce its width by 3 feet, the remaining area is $\frac{14}{15}$ths of what it would have been had its length been reduced by 3 feet. What are its length and breadth?

61. Multiply a^3-x^3 by a^2-x^2, and divide the result by $(a-x)^2$.

62. Simplify $(a-b)(b+c)(c+a)+(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)$, and find its value when $a=1$, $b=3$, $c=-2$.

63. Find, as a fraction in its lowest terms, the value of

$$\frac{1}{x^3-3x^2-15x+25}-\frac{1}{x^3+7x^2+5x-25}.$$

64. Solve the equations—

(*a*) $\sqrt{(x^2+a^2)}-\dfrac{b^2}{\sqrt{(x^2+a^2)}}=\dfrac{b^2-a^2}{a}$.

(*b*) $x+y=7$, $x^3+y^3=133$.

(*c*) $xy+\sqrt{x+y}=11$, $2xy-\sqrt{x+y}=13$.

65. What is the least integral multiplier which will make $17x^5-68x^4y+102x^3y^2-68x^2y^3+17xy^4$ a complete cube?

66. A rectangular plot of ground measures 42 acres, and its diagonal is 1243 yards long. Find its sides.

67. Factorize $a^3+3abc+b^3-c^3$, $x^4+y^4-11x^2y^2$, $x^2+2a(3x-bc)+8a^2-b^2c^2$.

68. Find the conditions in order that $a^2+nab+b^2$ may be a factor of $a^4+a^3b+a^2b^2+ab^3+b^4$.

69. Divide $(22x+3y-z)^2-(17x-2y+3z)^2$ by $5x+5y-4z$.

70. Find the highest common divisor of

$x^4+4ax^3+4a^2x^2-9a^4$, and $x^3+3ax^2+5a^2x+3a^3$.

71. Simplify—

(*a*) $\left(\frac{a+x}{a-x}-\frac{a-x}{a+x}\right)\div\left(\frac{a^2+x^2}{a^2-x^2}-\frac{a^2-x^2}{a^2+x^2}\right)$.

(*b*) $\frac{(a-b)^3+(b-c)^3+(c-a)^3}{(a-b)(b-c)(c-a)}$.

72. Solve the equations—

(*a*) $x+4+\sqrt{\frac{x+4}{x-4}}=\frac{12}{x-4}$.

(*b*) $2x+3y=11,\ 2x^2+3xy-4y^2=40$.

73. Reduce to its lowest terms—

$$\frac{x^4+ax^3-9a^2x^2+11a^3x-4a^4}{x^4-ax^3-3a^2x^2+5a^3x-2a^4}.$$

74. Divide $2a^3+7a^2b-9b^2(a+b)$ by $2a-3b$.

75. Simplify—

$$\{(a-b)^2+4ab\}^{\frac{1}{2}}\times\{(a+b)^2-4ab\}^{\frac{3}{2}}$$
$$\times\left\{\frac{a^4-b^4}{a-b}+2ab(a+b)\right\}^{\frac{2}{3}}.$$

76. Reduce to elementary factors—

$$4x^2y^2-(x^2+y^2-z^2)^2,\ a^4-14a^2b^2+b^4,$$
$$1-\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

77. Solve the equations—

(*a*) $\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x$.

(*b*) $\frac{x^2}{3m-2a}-\frac{x}{2}=\frac{m^2-4a^2}{4a-6m}$.

(*c*) $\sqrt[3]{x}+\sqrt[3]{y}=3,\ x+y=9$.

78. Multiply $x+2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$ by $x-2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$.

79. Reduce $\frac{ab(x^2-y^2)+xy(a^2-b^2)}{ab(x^2+y^2)+xy(a^2+b^2)}$.

80. Find the L. C. M. of x^3+x^2+x+1 and x^3-x^2+x-1.

81. Subtract $(x^2-5)^2$ from $(x-3)(x-1)(x+3)(x+1)$.

82. Divide

$$(a+b)^2-3(a+b)\sqrt{ab}+2ab \text{ by } (\sqrt{a}-\sqrt{b})^2.$$

83. Find the lowest common multiple of the two expressions x^2+6x+5 and x^3-x.

84. Reduce to its simplest form the fraction

$$\frac{m^3-6m^2+12m-8}{m^3-2m^2-4m+8}.$$

85. Divide the number 20 into two such parts that the square of the greater shall exceed the square of the less by 80.

86. Solve the equations—

$$(a)\ \frac{x-5}{x+5}-\frac{x+5}{x-5}=\frac{15}{4}.$$

$$(b)\ x^2-3xy+y^2=-1,\ x-y=3.$$

$$(c)\ x-y=2,\ x^4+y^4=706.$$

87. Prove that if any number be divided into two parts, the sum of the squares of these parts is least when the parts are equal.

88. If $\frac{a+b}{1-ab}+\frac{c+d}{1-cd}=0$, prove that

$$a+b+c+d=abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

89. Multiply $x^{\frac{5}{4}}-x^{\frac{3}{4}}+x^{\frac{1}{4}}-x^{-\frac{1}{4}}$ by $x^{\frac{3}{4}}+x^{\frac{1}{4}}$.

90. A railway train travels 288 miles at a certain average speed, and, if the speed were 4 miles an hour greater, would perform the journey in an hour less. What is the average speed of the train?

91. Resolve into elementary factors—

$$2b^2-5ab+2a^2+bx-ax-x^2.$$

$$2abc+a^2(b+c)+b^2(a+c)+c^2(a+b).$$

$$2(x^2y^2+x^2z^2+y^2z^2)-(x^4+y^4+z^4).$$

92. Solve the equations—

$$(a)\ \frac{3}{2(x^2-1)}-\frac{1}{4(x+1)}=\frac{1}{8}.$$

$$(b)\ 3x^4-7x^3+4x^2+x-1=0.$$

$$(c)\ x+y=7,\ x^3+y^3=91.$$

93. A number of persons undertake to erect a public building, contributing equally to the cost, which was £5000. Whilst it was in progress, 15 of the number became bankrupt, and 10 others refused to contribute to the deficiency thus occasioned. The deficiency being equally borne by the remainder increased the contribution of each by 15 per cent. What was the total number of subscribers?

94. Simplify $\dfrac{\frac{1}{ca}+\frac{1}{ab}-\frac{1}{bc}}{(b+c)^2-a^2}+\dfrac{\frac{1}{ab}+\frac{1}{bc}-\frac{1}{ca}}{(c+a)^2-b^2}+\dfrac{\frac{1}{bc}+\frac{1}{ca}-\frac{1}{ab}}{(a+b)^2-c^2}.$

95. If $4x^4+12x^3y+nx^2y^2+6xy^3+y^4$ be a perfect square, find n.

96. If $s=\frac{1}{2}(a+b+c)$, prove that

$$\frac{1-\dfrac{b^2+c^2-a^2}{2bc}}{1+\dfrac{b^2+c^2-a^2}{2bc}}=\frac{(s-b)(s-c)}{s(s-a)}.$$

97. Find the G. C. M. of x^m-y^m and x^n-y^n.

98. Expand the continued product—

$$a^2x^2(1+a^{-1}x)(1+a^{-1}x-a^{-1})(1-x^{-1}a)(1-x^{-1}a+x^{-1}).$$

99. Solve the equations—

$$(a)\ \sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}.$$

$$(b)\ 8x^{\frac{3}{4}}+81=18x^{\frac{1}{2}}+45x^{\frac{1}{4}}.$$

$$(c)\ (x+a)\left(1+\frac{1}{x^2+a^2}\right)+\sqrt{2ax}\left(1-\frac{1}{x^2+a^2}\right)=2.$$

100. A horse is sold for £24, and the number ex-

pressing the profit per cent. expresses also the cost price in pounds of the horse. What did the horse cost?

101. Show that 3 is a root of the equation—

$$\sqrt{x^2-3x+4}=\frac{3+\sqrt[3]{x-2}}{x^2-7}.$$

102. Find the L. C. M. of $5x^2-15x+10$, $6x^2-6x-12$, and $12x^2-12$.

103. Reduce to its simplest form—

$$\frac{x^4-5x^2+4}{x^2+1}\times\frac{\frac{1}{x}-\frac{1}{x+2}}{1-\frac{1}{x^2}}\div\frac{2-\frac{4}{x}}{1+\frac{1}{x^2}}.$$

104. Simplify

$$(a+b+c)(a^{-1}+b^{-1}+c^{-1})-a^{-1}b^{-1}c^{-1}(b+c)(c+a)(a+b).$$

105. Multiply $x^2+x(a^{\frac{1}{2}}-b^{\frac{1}{2}})-a^{\frac{1}{2}}b^{\frac{1}{2}}$ by $x^2-x(a^{\frac{1}{2}}-b^{\frac{1}{2}})-a^{\frac{1}{2}}b^{\frac{1}{2}}$.

106. Solve the equations—

(*a*) $x^2+\frac{1}{x^2}-\frac{25}{6}\left(x-\frac{1}{x}\right)+2=0.$

(*b*) $2x^2-3xy+y^2=24$, $x^2-2xy+y^2=9$.

(*c*) $(x^2-y^2)(x-y)=16xy$, $(x^4-y^4)(x^2-y^2)=640x^2y^2$.

107. The product of the sum and difference of a number and its reciprocal is $3\frac{3}{4}$. Find the number.

108. Divide $(a^3-9a^2b+23ab^2-15b^3)(a-7b)$ by $a^2-8ab+7b^2$.

109. Simplify the expression—

$$\frac{x^6+y^6}{x^6-y^6}\times\frac{x-y}{x+y}\div\frac{x^4-x^2y^2+y^4}{x^4+x^2y^2+y^4},$$

and find its value when $x=y$.

110. Simplify $\frac{x^3-4x^2+5x-2}{x^2-1}$, and find the factors of $x^3+3axy+y^3-a^3$.

111. Solve the equations—

(*a*) $\dfrac{x}{\sqrt{1-x}}+\dfrac{\sqrt{1-x}}{x}=\dfrac{13}{6}$.

(*b*) $x^2y+xy^2=180$, $x^3+y^3=189$.

(*c*) $x+y+z=6$, $x^2+y^2+z^2=38$, $x^3+y^3+z^3=216$.

112. Two men start at the same time to meet each other, from towns which are 25 miles apart. If one takes 18 minutes longer than the other to walk a mile, and they meet in 5 hours, how fast does each walk?

113. Prove that any trinomial is a complete square, if the square of the middle term is equal to four times the product of the first and last terms.

114. If $yz=\sqrt{ay^2-a^2}$, and $xy=\sqrt{ax^2-a^2}$, find the value of $\frac{1}{z}\sqrt{az^2-a^2}$ in terms of x.

115. Find the value of a when the fraction

$$\frac{x^3+(2a-1)x^2+x+1}{x^3+(2a+1)x^2+3x+1}$$

admits of reduction, and reduce it.

116. Find the G. C. M. of $x^4+3x^3-7x^2+3x+28$, and $x^3+2x^2-11x+20$. Find also their L. C. M.

117. Simplify—

(*a*) $\dfrac{7}{2(x-2)}-\dfrac{3}{x+2}+\dfrac{12}{4-x^2}$.

(*b*) $\dfrac{1-\dfrac{1}{x}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}}\div\dfrac{x-\dfrac{1}{x}}{1+x+x^2}$.

118. Solve the equations—

(*a*) $(x-1)(x-2)(x+3)-(x+1)(x+2)(x-4)=54$.

(*b*) $x^2-xy=44$, $xy-y^2=28$.

119. Divide $(x^2-p^2)(x^4+p^2x^2+p^4)$ by $(x^2+px+p^2)(x-p)$.

120. The difference of two numbers is 6, and the difference of their squares exceeds their sum by 70. Find the numbers.

121. Find the G. C. M. and L. C. M. of x^4-y^4, x^3+y^3, $x^3y+x^2y^2+xy^3+y^4$.

122. Reduce to their simplest forms—

$$\frac{5\sqrt{2}-\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} \text{ and } \sqrt{31-4\sqrt{21}}.$$

123. Find the continued product of $x^2+x\sqrt{2}+2$, $x^2-x\sqrt{2}+2$, $x^2+x\sqrt{6}+2$, and $x^2-x\sqrt{6}+2$.

124. Express $x^4+\frac{1}{64}$ as the product of two real quadratic factors.

125. Find a number which shall exceed its square root by 110.

126. Divide $(3a^2-6ab+8b^2)^2-4(a^2-ab+2b^2)^2$ by $5a^2-8ab+12b^2$.

127. Reduce to their simplest form—

(*a*) $1+x^2+x^4-\dfrac{x-x^6}{1-x^2}$.

(*b*) $\left(x+\dfrac{1}{x}\right)\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2-\left(x-\dfrac{1}{x}\right)^2$.

128. Express $(x^3-3x^2)^2-(3x-1)^2$ as the product of five real integral factors.

129. If $x=5+\sqrt{3}$, show that $x^2+\dfrac{484}{x^2}=56$.

130. Solve the equations—

(*a*) $\dfrac{21x^3-16}{3x^2-4}=7x+5$.

(*b*) $27x^3-100x^2+225x=35x^2+117$.

ANSWERS TO EXERCISES.

Exercise 1.

1. $5ab^2(2a+3b-5c)$.
2. $11x^2y^3z^4(5xy-7xz+13yz)$.
3. $6am(5a-3m-am)$.
4. $5x^3(x^3-3x^2+7x-15)$.
5. $(a+b)(c+d)$.
6. $(2a+3b)(x-y)$.
7. $(2d-5e)(a+b+c)$.
8. $(3x^2-5y^2)(m^2+3)$.
9. $(2x^2-5y^2)(3c-a-2b)$.
10. $(a^2-2ab+7b^2)(m^2-3mn+5n^2)$.
11. $(a-b^3+c^5)(x^2-5y^2+11z^2)$.
12. $8m^2n^2(a-b)(x+3y)$.

Exercise 2.

1. $(2a+3b)(2a-3b)$.
2. $\left(\frac{x}{4}+\frac{4}{x}\right)\left(\frac{x}{4}-\frac{4}{x}\right)$.
3. $(6xy+1)(6xy-1)$.
4. $(x+y+xy)(x+y-xy)$.
5. $(m-n+p)(m-n-p)$.
6. $(m+n-p)(m-n+p)$.
7. $(x^2+9)(x+3)(x-3)$.
8. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.
9. $(x+y+z)(x+y-z)$.
10. $(c+a-b)(c-a+b)$.
11. $(a+b+c-d)(a+b-c+d)$.
12. $\frac{(x+y+z)(y+z-x)}{2yz}$.
13. $9x^2-49$.
14. $\frac{a^2}{4}-\frac{4}{a^2}$.
15. $9a^2b^2c^2-25$.
16. $a^2+2ab+b^2-a^2c^2$.
17. $81x^4-16y^4$.
18. $a^2-1-\frac{2}{a}-\frac{1}{a^2}$.
19. $a^2-2ac-b^2-2bd+c^2-d^2$.
20. $x^2+1+\frac{1}{x^2}$.
21. $a^2-\frac{2a}{c}-b^2+\frac{1}{c^2}$.
22. $\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}+\frac{2}{ac}$.

Exercise 3.

1. $(2a-3b)(4a^2+6ab+9b^2)$.
2. $(5p-1)(25p^2+5p+1)$.
3. $(\frac{4}{3}x-\frac{3}{4}y)(\frac{16}{9}x^2+xy+\frac{9}{16}y^2)$.
4. $-2y(3x^2+y^2)$.
5. $(a+b-8)(a^2+2ab+b^2+8a+8b+64)$.
6. $d(3c^2-3cd+d^2)$.
7. $\dfrac{xz-xy-y^2}{yz}\cdot\dfrac{x^2z^2+x^2y^2+x^2yz+2xy^3+xy^2z+y^4}{y^2z^2}$.
8. $\dfrac{5-m-n}{5}\cdot\dfrac{25+5m+5n+m^2+2mn+n^2}{25}$.
9. $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}\right)$. 10. $\left(\dfrac{a}{2}-2\right)\left(\dfrac{a^2}{4}+a+4\right)$.

Exercise 4.

1. $(2a+3b)(4a^2-6ab+9b^2)$.
2. $(5p+1)(25p^2-5p+1)$.
3. $(a+b)(a^2-ab-3ac+b^2+3bc+3c^2)$.
4. $(2x-y)(x^2-xy+y^2)$. 5. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
6. $(x+y)(x^2-xy+y^2)(x^6-x^3y^3+y^6)$.
7. $\dfrac{a+2b}{c}\cdot\dfrac{a^2+ab+b^2}{c^2}$.
8. $\dfrac{3+a+b}{3}\cdot\dfrac{9-3a-3b+a^2+2ab+b^2}{9}$.
9. $\left(x+\dfrac{1}{x}\right)\left(x^2-1+\dfrac{1}{x^2}\right)$. 10. $\left(\dfrac{a^4}{2}+\dfrac{2}{a^4}\right)\left(\dfrac{a^8}{4}-1+\dfrac{4}{a^8}\right)$.

Exercise 5.

1. $(a^2+3a+9)(a^2-3a+9)$.
2. $(4m^2+6mn+9n^2)(4m^2-6mn+9n^2)$.
3. $(3a^2+b^2)(a^2+3b^2)$.
4. $(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$.
5. $(1+x+x^2)(1-x+x^2)$.
6. $(a^2+7a+14)(a^2+5a+8)$.

7. $9(a^2+ab+3b^2)(a^2-ab+3b^2)$.
8. $(5x^2+xy-3y^2)(5x^2-xy-3y^2)$.
9. $(m^2+4mn+8n^2)(m^2-4mn+8n^2)$.
10. $\left(a^2+\frac{2a}{b}+\frac{2}{b^2}\right)\left(a^2-\frac{2a}{b}+\frac{2}{b^2}\right)$.
11. $(7x^2+11xy+9y^2)(7x^2-11xy+9y^2)$.
12. $(11a^2+27ab-3b^2)(11a^2-27ab-3b^2)$.

Exercise 6.

1. $(x-y)(x+y)(x^2-xy+y^2)(x^2+xy+y^2)$.
2. $(x^5-2)(x^5+2)(x^{10}-2x^5+4)(x^{10}+2x^5+4)$.
3. $\left(\frac{a^2}{2}-\frac{1}{5}\right)\left(\frac{a^2}{2}+\frac{1}{5}\right)\left(\frac{a^4}{4}-\frac{a^2}{10}+\frac{1}{25}\right)\left(\frac{a^4}{4}+\frac{a^2}{10}+\frac{1}{25}\right)$.
4. $b^6(a-c)(a+c)(a^2-ac+c^2)(a^2+ac+c^2)$.
5. $(2a-3)(a^2-3a+3)(3a^2-9a+7)$.
6. $-\frac{3a+b}{a+b}\cdot\frac{2a}{a+b}\cdot\frac{4a^2+2ab+b^2}{(a+b)^2}\cdot\frac{4a^2+6ab+3b^2}{(a+b)^2}$.
7. $(5x-3y)(5x+3y)(25x^2-15xy+9y^2)(25x^2+15xy+9y^2)(5x^2+9y^2)(625x^4-225x^2y^2+81y^4)$.
8. $\frac{2a}{b-c}\cdot\frac{2b}{b-c}\cdot\frac{3a^2+b^2}{(b-c)^2}\cdot\frac{a^2+3b^2}{(b-c)^2}$.

Exercise 7.

1. $(x+2)(x+13)$.
2. $(a+3b)(a+12b)$.
3. $(x+y)(x+14y)$.
4. $(x+1)(x+299)$.
5. $(m+48n)(m+2n)$.
6. $(ab+3cd)(ab+14cd)$.
7. $(a+b+2)(a+b+9)$.
8. $(x+6y+5z)(x+19y+18z)$.
9. $(a+18)(a+10)$.
10. $2(8x-7y)(5x-3y)$.
11. $(a+b)(a^2-ab+b^2)(a^3+18b^3)$.
12. $(x+2y)(x^2-2xy+4y^2)(x^3+23y^3)$.

Exercise 8.

1. $(x-3)(x-62)$.
2. $(p-5q)(p-28q)$.
3. $(a^2-3b^3)(a^2-48b^3)$.
4. $(x^4-11)(x^4-84)$.
5. $(mn-25pq)(mn-48pq)$.
6. $(a^4b^5-14c^2d^3)(a^4b^5-15c^2d^3)$.
7. $(x-11)(x-7)$.
8. $(3a-10)(18a-55)$.
9. $(a+3b)(a-3b)(a^2-17b^2)$.
10. $(x-2y)(x+2y)(x^2+4y^2)(x^2-5y^2)(x^2+5y^2)$.
11. $(m-1)(m^2+m+1)(m^3-13)$.
12. $(a^3-2b^3)(a^3-3b^3)(a^6+2a^3b^3+4b^6)(a^6+3a^3b^3+9b^6)$.

Exercise 9.

1. $(x+11y)(x-7y)$.
2. $(x-15y)(x+3y)$.
3. $(a-19)(a+11)$.
4. $(a+17b)(a-3b)$.
5. $(p+19q)(p-5q)$.
6. $(ab-63cd)(ab+9cd)$.
7. $(a-3b)(a+3b)(a^2+17b^2)$.
8. $(x-3y)(x^2+3xy+9y^2)(x^3+7y^3)$.
9. $(x+2y)(x^2-2xy+4y^2)(x^3-35y^3)$.
10. $(a-5b)(a+3b)(a^2-3ab+9b^2)(a^2+5ab+25b^2)$.
11. $2(11a-10b)(3b-a)$.
12. $(1-11a+11b)(1+10a-10b)$.

Exercise 10.

1. $(16x+27)(x+1)$.
2. $(8x-27y)(2x-y)$.
3. $(4a+b)(4a-27b)$.
4. $(16x-y)(x+27y)$.
5. $(2m-27n)(8m-n)$.
6. $(4+3x)(4-9x)$.
7. $(16xyz-9)(xyz+3)$.
8. $(8a+9b)(2a+3b)$.
9. $(16a+16b+3c+3d)(a+b+9c+9d)$.
10. $(8x-3y-23)(2x-9y+19)$.
11. $(5x+12y)(8x-7y)$.
12. $2(20a+bc)(a-42bc)$.
13. $(a+b+2c)(a+b-c)$.
14. $(ax+d)(bx-c)$.
15. $(mx-nx-3m)(x+1)$.
16. $(mx-nx+m)(x-3)$.
17. $(4x-y)(4x+y)(2x-3y)(2x+3y)$.
18. $(x+y)(2x+3y)(x^2-xy+y^2)(4x^2-6xy+9y^2)$.
19. $(5x-y)(25x^2+5xy+y^2)(7x^3+19y^3)$.
20. $(3x-5y)(3x+5y)(17x^2+11y^2)$.
21. $(a^2+b^2)(a^2-2b^2)(a^4-a^2b^2+b^4)(a^4+2a^2b^2+4b^4)$.

Exercise 11.

1. $(6x+y+3z)(8x+8y+3z)$.
2. $(9a-4b+12c)(8a-6b-9c)$.
3. $(8x+4y-11z)(3x-5y+7z)$.
4. $(3x-y-z)(9x-7y-9z)$.
5. $(12a+12b+c)(a-12b-12c)$.
6. $(11x+12y-15z)(x-y+z)$.
7. $(4mn+5np-4mp)(mn+5np-2mp)$.
8. $(3a^2-2b^2-1)(2a^2-3b^2-1)$.
9. $(7a-9b)(5a-3b+2c)$.
10. $(a-2c)(a+2c)(6a^2-7b^2+11c^2)$.
11. $(x+y)(x^2-xy+y^2)(4x^3-9y^3+7z^3)$.
12. $(13-9y^2)(7-3x^2-5y^2)$.

Exercise 12.

1. $(x-y)(7x^2-9xy+6y^2)$.
2. $(a-1)(a-2)(8a+13)$.
3. $(x-y)(13x^2-18xy-5y^2)$.
4. $(x-y)^5$.
5. $(m-1)^2(5m+9)(7m-3)$.
6. $(x-y)(2x+3y)(2x-3y)$.

Exercise 13.

1. $(x+y)(3x^2+5xy+17y^2)$.
2. $(a+b)(a-5b)(2a+b)$.
3. $(1+m)(4-6m+3m^2)$.
4. $(x+y)^2(3x-5y)(2x-7y)$.
5. $(a-1)(a+1)(2a^2-7a+20)$.
6. $(x+1)^2(x-1)^2(3x-2)$.

Exercise 14.

1. $(x-11)(5x^2-7x+9)$.
2. $(a-7b)(a+8b)(3a-7b)$.
3. $(x+5y)(2x-3y)(4x+y)$.
4. $(a+2)(a-3)(8a-3)(5a-2)$.
5. $(x-2y)(2x+y)(2x-3y)(2x-y)(2x+3y)$.
6. $(x-7)(x-1)^3(x+1)^3$.

Exercise 15.

1. $x^7+x^6y+x^5y^2+x^4y^3+x^3y^4+x^2y^5+xy^6+y^7$.
2. $27a^6-9a^4+3a^2-1+\frac{2}{3a^2+1}$.
3. $a^2+2ab+4b^2$.
4. $x^8+x^6y^2+x^4y^4+x^2y^6+y^8+\frac{2y^{10}}{x^2-y^2}$.
5. $625a^4b^4-375a^3b^3+225a^2b^2-135ab+81$.
6. $a^7+a^5+a^3+a+\frac{1}{a}+\frac{1}{a^3}+\frac{1}{a^5}+\frac{1}{a^7}+\frac{2}{a^7(a^2-1)}$.
7. $a^6-a^2+\frac{1}{a^2}-\frac{1}{a^6}$.
8. $1-2a+4a^2-8a^3+16a^4-32a^5+64a^6-\frac{256a^7}{1+2a}$.
9. A factor of the form $x-y$, viz. $(a+b)-(a-b)=2b$.
10. A factor of the form $x+y$, viz. $(a+b)+(a-b)=2a$.
11. A factor of the form $x+y$, viz. x^4+1.
12. Both, viz. $a^2b^3c^2(a^2+c^2)$, and $a^2b^3c^2(a^2-c^2)$.

Exercise 16.

1. $(x+y-z)(x^2+y^2+z^2-xy+xz+yz)$.
2. $(1-x+3y)(1+x^2+9y^2+x-3y+3xy)$.
3. $(a-2b-3c)(a^2+4b^2+9c^2+2ab+3ac-6bc)$.
4. $(5a+3b-2)(25a^2+9b^2+4-15ab+10a+6b)$.
5. $(x+2y+2z)(x^2+4y^2+4z^2-2xy-2xz-4yz)$.
6. $(3a-b+2c)(9a^2+b^2+4c^2+3ab-6ac+2bc)$.

Exercise 17.

1. $4x^2+20xy+25y^2-12xz-30yz+9z^2$.
2. $9x^2-42xy+49y^2-54xz+126yz+81z^2$.
3. $216x^3-343y^3$.
4. $729a^3+1331b^3$.
5. $125a^6+27$.
6. $343x^9-8y^6$.

7. $a^4 - 1$.
8. $243a^5 - 32b^5$.
9. $36x^4 - 97x^2y^2 + 36y^4$.
10. $64x^{12} - 1$.
11. $a^5 - a^3b^2 + a^2b^3 - b^5$.
12. $(a+b)^4x^4 - 2(a+b)^2x^2 + 1$.
13. $x^3 + 3(a+b)x^2 + 3(a+b)^2x + (a+b)^3$.
14. $x^3 + 6ax + 8a^3 - 1$.

Exercise 18.

1. $a^2 + 2ab + b^2$.
2. $a^8 + a^4b^4 + b^8$.
3. $3x^2 + 5xy + 2y^2$.
4. $a^2 - 5ab + 25b^2$.
5. $x^2 - px + q$.
6. $a + b - c - d$.
7. $x^2 + y^2 - 1$.
8. $m^2 - (a+c)m + ac$.

Exercise 19.

1. $a - b$.
2. $a^2 - ab + b^2$.
3. $x^2 + y^2$.
4. $2x - 3y$.

Exercise 20.

1. $a + 5b$.
2. $1 - 8x$.
3. $5x + 3y$.
4. $11a - 11b$.
5. $4a - 7b$.
6. $8x - 1$.

Exercise 21.

1. $5x - 9y + 3z$.
2. $2x^2 - 5y^2 + 1$.
3. $7a + 8b - 6$.
4. $xy - 2xz + 3yz$.

Exercise 22.

1. $5x - 7y$.
2. $2x - 5$.
3. $2a - 9b$.
4. $5m - 11n$

Exercise 23.

1. $2a - 3$.
2. $5x - 9y$.
3. $4x - 3$.
4. $9x - 5y$.

Exercise 24.

1. $132a^3bx^3y^3$.
2. $2400a^3b^2c^3$.
3. $36a^2b^2c^2(x^2 - y^2)$.
4. $60a^2(a^2 - b^2)(a^3 - b^3)$.
5. $a^2b^2(a^6 - b^6)$.
6. $12(x^2 - 9)^2$.
7. $(x - 2)(x - 3)(x - 4)$.
8. $(4x - 7y)^2(8x + 9y)^2(11x - 3y)$.
9. $a^6 - b^6$.
10. $(x + 3y)(3x - y)(x - 3y)^2$.

Exercise 25.

1. $\dfrac{8x + 3}{8x + 4}$.
2. $\dfrac{a^2 + 2ab + 4b^2}{(a - 2b)^2}$.
3. $\dfrac{4x + 7y}{5x^2 - 2xy + y^2}$.
4. $\dfrac{x^2 + 3x + 8}{8x^2 + 3x + 1}$.
5. $\dfrac{a + b - c - d}{a - b - c + d}$.
6. $\dfrac{x - y + z}{x + y + z}$.

Exercise 26.

1. $\dfrac{2}{x - 3y}$.
2. $\dfrac{-1}{(x - 1)(x - 3)}$.
3. 3.
4. $\dfrac{a}{(a - 2x)^2}$.
5. $2a^{\frac{1}{3}}b^{\frac{1}{3}}$.
6. $\dfrac{m + nx}{n + mx}$.
7. 1.
8. $2(a - x)$.
9. $\dfrac{2}{(a^2 - 1)^2}$.
10. $\dfrac{4(y - z)}{x + y - z}$.
11. $\dfrac{1}{abc}$.
12. $\dfrac{24b^4}{a(a^2 - b^2)(a^2 - 4b^2)}$.

Exercise 27.

1. $-\dfrac{(a^2+b^2)(a^2-ab+b^2)}{(a^2-b^2)(a^2+ab+b^2)}$.
2. $\dfrac{(2a^2+x^2)(a-x)}{a^2x}$.
3. $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}$.
4. $\dfrac{x^2-1}{y^2-1}$.
5. $\dfrac{(a^2-b^2)^4}{a^5}$.
6. $\dfrac{2x(x+y)}{y^2}$.
7. 1.
8. $10\frac{2}{3}$.

Exercise 28.

1. $9, -3$.
2. $1, -\frac{2}{9}$.
3. $2, \frac{1}{2}$.
4. $3\frac{1}{2}, 6$.
5. $5, -1\frac{1}{4}$.
6. $2a, b-a$.
7. a, b.
8. $2, \frac{47}{9}$.
9. $9, -3\frac{3}{5}$.
10. $a, \dfrac{b^2-a^2}{a}$.

Exercise 29.

1. $3, 27, 1, 29$.
2. $5, 21$.
3. $-5, 0$.
4. $1, \frac{1}{2}$.
5. $2, 3\frac{1}{2}$.
6. $\sqrt[3]{5}, \sqrt[3]{-3}$.
7. $a, 0$.
8. $\pm 2c\sqrt{-2}, \pm c\sqrt{-15}$.

Exercise 30.

1. $1, -\frac{1}{2}, -3$.
2. $\dfrac{\pm\sqrt{3}}{2}, \dfrac{1\pm\sqrt{7}}{4}$.
3. $0, \pm a\sqrt{-2}$.
4. $1, 2, \frac{2}{3}$.
5. $1, -1, \frac{2}{7}$.
6. $1, \dfrac{7\pm\sqrt{-59}}{3}$.

Exercise 31.

1. $x - 2y$.
2. $7x + 8y$.
3. $5a - 17b$.
4. $3a^2 - 7$.
5. $4x - 11$.
6. $x^2 + 7x - 5$.

Exercise 32.

1. $x + 2y$.
2. $a^2 - b^2$.
3. $1 - 7x$.
4. $2a^2 - 11b^3$.
5. $2b$.
6. $2a + 5$.

Exercise 33.

1. 1·641.
2. 3·236.
3. 7·194.
4. 19·485.
5. 6.
6. ·270.
7. ·548.
8. 5·471.

Exercise 34.

1. $5 + \sqrt{7}$.
2. $3 - \sqrt{2}$.
3. $\sqrt{7} + \sqrt{5}$.
4. $\sqrt{7} - \sqrt{3}$.
5. $3\sqrt{2} - 2\sqrt{3}$.
6. $3\sqrt{11} + 1$.

Miscellaneous Questions.

1. $\dfrac{3x+9}{x^4 - x^3 + 6x^2 - 6x + 6}$.
2. 1·129.
3. (*a*) 4, 1; 1, 4. (*b*) 5, 4. (*c*) 5.
4. ·755.
5. $x^3 - 9x^2 + 26x - 24$; $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.
6. $x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$; $x + y$.

7. $a^4 - 2a^2b^2 + b^4$; $a^2 - 2ab + b^2 - c^2 - 2cd - d^2$.

8. $\frac{y}{x}$.

10. $\frac{(x+1)^2 (x-a)^2}{(a+b)(b-x)}$.

11. $2(x^2y^2 + x^2z^2 + y^2z^2) - (x^4 + y^4 + z^4)$.

14. $\frac{(b+c+a)(b+c-a)}{2bc}$; $(7x-8y)(5x-3y)$;
$(a^2 - 2b^2)(a^4 + 2a^2b^2 + 4b^4)$;
$(2x+3)(4x^2 - 7x + 5)$.

15. 8, 5.

16. 1·77.

17. $(4k^2 - y^2)(4k^2 - 9y^2)$.

19. 3·525.

20. (*a*) $\frac{4a}{5}$, 0; (*b*) $1 \pm \frac{1}{\sqrt{a+1}}$; (*c*) 1.

21. $\sqrt{5} + \sqrt{3} = 1{\cdot}254$.

22. $315(a+x)(a^4 - x^4)$.

23. $2(a-b)$.

24. $\frac{3a^4 + 6a^2x^2 - x^4}{(a^2 - x^2)^2}$.

25. $\frac{4x-5}{(3x-4)(4x-3)}$.

27. $\frac{(3x+1)(2x+1)}{5x+1}$.

28. 405, 805 yards.

30. $(1+x)(1-x)(1-2x)(1-3x)(1+4x)(1-4x)$.

31. (*a*) $-\frac{14a}{5}$; (*b*) 12, 3.

33. $1800a^3x^8y^3$.

34. $\frac{(x-2)(x+5)}{(x+3)(x-4)}$.

35. $3y(3x+2y)$.

36. $(x+y+\sqrt{xy})(x+y-\sqrt{xy})$, $(x+y-z)(x-y+z)$,
$4xy(x+y)(x-y)$.

37. 250.

38. $(x^3+y^3)(x^4+y^4) = 4$.

39. (*a*) $\frac{1}{5}$; (*b*) ± 6, ± 5; (*c*) $\pm \frac{\sqrt{2}}{\sqrt[4]{3}}$, $\pm \sqrt{\frac{-2}{\sqrt{3}}}$.

42. (*a*) 6, $2\frac{1}{2}$; (*b*) ± 4, ± 2, ± 5; (*c*) $6 \mp 2\sqrt{6}$, $6 \pm 2\sqrt{6}$,
or $x = y = 0$ or 10.

43. $\sqrt{22} + \sqrt{17}$.

44. $(m+n-p-q)(m-n+p-q)$, $(x^2+x+1)(x^2-x+1)$
$(x^4 - x^2 + 1)$, $a^{n+1}b^{-1}(a^n + b^{-1})(a^n - b^{-1})$,
$(x-y+z)(x^2+y^2+z^2+xy-xz-2yz)$.

45. $\pm \frac{1}{\sqrt{3}}$, $\pm \sqrt{3}$.

46. $-2x+y$.

47. $x = y = z = \pm a$.

49. ·618, ·382.

51. $\frac{1}{2} \pm \sqrt{\frac{1}{4} + \sqrt[n]{\frac{a}{b}}}$.

53. (*a*) $\frac{x+1}{4}$; (*b*) $\frac{12}{(x^2-9)(x^2-4)}$.

54. (*a*) $\frac{a+b}{a}$, -1; (*b*) 2, -1; -1, 2.

55. 60.

56. $\frac{(y+z+a)(y+z-a)(y-z+a)(z-y+a)}{4y^2z^2}$,

$(a-1)(a+1)(a-3)(a+3)$, $(x-1)(2x+1)$, $(ax-c)(bx+c)$.

59. $\frac{4}{5}$, $2\frac{2}{5}$; $2\frac{2}{5}$, $1\frac{1}{5}$.

60. 21, 15 feet.

61. $a^3+2a^2x+2ax^2+x^3$.

62. $(a-b)(a-c)(b-c) = -30$.

63. $\frac{10}{(x-5)(x+5)(x^2+2x-5)}$.

64. (*a*) 0, $\pm\frac{\sqrt{b^4-a^4}}{a}$; (*b*) 5, 2; 2, 5; (*c*) 8, 1; 1, 8.

65. $289x^2(x-y)^2$.

66. 1232, 165 yards.

67. $(a+b-c)(a^2+b^2+c^2-ab+ac+bc)$,

$(x^2+3xy-y^2)(x^2-3xy-y^2)$, $(x+4a+bc)(x+2a-bc)$.

68. $na+b=0$.

69. $39x+y+2z$.

70. $x^2+2ax+3a^2$.

71. (*a*) $\frac{a^2+x^2}{ax}$; (*b*) 3.

72. (*a*) ± 5; (*b*) 4, $\frac{211}{16}$; 1, $-\frac{41}{8}$.

73. $\frac{x+4a}{x+2a}$.

74. $a^2+5ab+3b^2$.

75. $(a^2-b^2)^3$.

76. $(x+y+z)(x+y-z)(x-y+z)(y+z-x)$,

$(a^2+4ab+b^2)(a^2-4ab+b^2)$; $\frac{(a+c+d-b)(b+c+d-a)}{2(ab+cd)}$.

77. (*a*) 0, $\pm\sqrt{3}$; (*b*) $m-2a$, $\frac{m+2a}{2}$; (*c*) 8, 1; 1, 8.

78. $x^2 - 4y + 9z^{\frac{2}{3}} + 6xz^{\frac{1}{3}}$. **79.** $\dfrac{ax - by}{ax + by}$.

80. $x^4 - 1$. **81.** -16.

82. $a + b - \sqrt{ab}$. **83.** $x(x+5)(x^2-1)$.

84. $\dfrac{m-2}{m+2}$. **85.** 12, 8.

86. (*a*) 3, $-8\frac{1}{3}$; (*b*) 5, -2; 2, -5;
(*c*) 5, -3, $1 \pm \sqrt{-22}$; 3, -5, $-1 \pm \sqrt{-22}$.

89. $x^2 - 1$. **90.** 32 miles.

91. $(2a+b+x)(a+2b-x)$, $(a+b)(a+c)(b+c)$; see Question 11.

92. (*a*) 3, -5; (*b*) 1, 1, $\dfrac{1 \pm \sqrt{13}}{6}$; (*c*) 4, 3; 3, 4.

93. 125. **94.** $\dfrac{3}{abc(a+b+c)}$.

95. 13.

97. If m is a multiple of n, $x^n - y^n$.
If n is a multiple of m, $x^m - y^m$.
If m and n are both even, $x^2 - y^2$.
In all other cases, $x - y$.

98. $x^4 - (2a^2 - 2a + 1)x^2 + a^2(a^2 - 2a + 1)$.

99. (*a*) 1. (*b*) $(-\frac{9}{4})^4$, $(\frac{3}{2})^4$. (*c*) $1 \pm \sqrt{2a - a^2}$.

100. £20. **102.** $60(x-2)(x^2-1)$.

103. 1. **104.** 1.

105. $x^4 - (a+b)x^2 + ab$.

106. (*a*) 2, $-\frac{1}{2}$, 3, $-\frac{1}{3}$. (*b*) ± 5, ± 2. (*c*) 0, 9, 3; 0, 3, 9.

107. 2. **108.** $a^2 - 8ab + 15b^2$. **109.** $\dfrac{x^2+y^2}{(x+y)^2}$, $\frac{1}{2}$.

110. $\dfrac{(x-1)(x-2)}{x+1}$; $(x+y-a)(x^2+y^2+a^2-xy+ax+ay)$.

111. (*a*) $\frac{3}{4}$, -3, $\dfrac{\pm 2\sqrt{10} - 2}{9}$. (*b*) 4, 5; 5, 4. (*c*) 1, -1, 6

112. 18 min. and 36 min. per mile.

114. $\pm x$. **115.** 1; $\dfrac{x^2+1}{(x+1)^2}$.

116. $x^2 - 3x + 4$; $(x^2 - 3x + 4)(x^2 + 6x + 7)(x + 5)$.

117. (*a*) $\frac{1}{2(x-2)}$; (*b*) $\frac{x^2}{x+1}$.

118. (*a*) 5, -8; (*b*) ± 11, ± 7.

119. $x^3 + p^3$. **120.** 10, 4.

121. $x + y$; $y(x^2 - xy + y^2)(x^4 - y^4)$.

122. $\frac{24 + 7\sqrt{6}}{6}$; $2\sqrt{7} - \sqrt{3}$.

123. $x^8 + 4x^4 + 16$. **124.** $(x^2 - \frac{1}{2}x + \frac{1}{8})(x^2 + \frac{1}{2}x + \frac{1}{8})$.

125. 121. **126.** $(a - 2b)^2$.

127. (*a*) $\frac{1}{1+x}$. (*b*) $4 - 2x - \frac{2}{x}$.

128. $(x-1)(x-1)(x-1)(x+1)(x^2 - 4x + 1)$.

130. (*a*) 2, $-\frac{2}{15}$. (*b*) 1, $\frac{1}{8}(6 \pm \sqrt{-3})$.

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Exercise 23.

1. $2a - 3$.
2. $5x - 9y$.
3. $4x - 3$.
4. $9x - 5y$.

Exercise 24.

1. $132a^3bx^3y^3$.
2. $2400a^3b^2c^3$.
3. $36a^2b^2c^2(x^2 - y^2)$.
4. $60a^2(a^2 - b^2)(a^3 - b^3)$.
5. $a^2b^2(a^6 - b^6)$.
6. $12(x^2 - 9)^2$.
7. $(x - 2)(x - 3)(x - 4)$.
8. $(4x - 7y)^2(8x + 9y)^2(11x - 3y)$.
9. $a^6 - b^6$.
10. $(x + 3y)(3x - y)(x - 3y)^2$.

Exercise 25.

1. $\dfrac{8x + 3}{8x + 4}$.
2. $\dfrac{a^2 + 2ab + 4b^2}{(a - 2b)^2}$.
3. $\dfrac{4x + 7y}{5x^2 - 2xy + y^2}$.
4. $\dfrac{x^2 + 3x + 8}{8x^2 + 3x + 1}$.
5. $\dfrac{a + b - c - d}{a - b - c + d}$.
6. $\dfrac{x - y + z}{x + y + z}$.

Exercise 26.

1. $\dfrac{2}{x - 3y}$.
2. $\dfrac{-1}{(x - 1)(x - 3)}$.
3. 3.
4. $\dfrac{a}{(a - 2x)^2}$.
5. $2a^{\frac{1}{3}}b^{\frac{1}{3}}$.
6. $\dfrac{m + nx}{n + mx}$.
7. 1.
8. $2(a - x)$.
9. $\dfrac{2}{(a^2 - 1)^2}$.
10. $\dfrac{4(y - z)}{x + y - z}$.
11. $\dfrac{1}{abc}$.
12. $\dfrac{24b^4}{a(a^2 - b^2)(a^2 - 4b^2)}$.

Exercise 27.

1. $-\dfrac{(a^2+b^2)(a^2-ab+b^2)}{(a^2-b^2)(a^2+ab+b^2)}$.
2. $\dfrac{(2a^2+x^2)(a-x)}{a^2x}$.
3. $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}$.
4. $\dfrac{x^2-1}{y^2-1}$.
5. $\dfrac{(a^2-b^2)^4}{a^5}$.
6. $\dfrac{2x(x+y)}{y^2}$.
7. 1.
8. $10\frac{2}{3}$.

Exercise 28.

1. $9, -3$.
2. $1, -\frac{2}{9}$.
3. $2, \frac{1}{2}$.
4. $3\frac{1}{2}, 6$.
5. $5, -1\frac{1}{4}$.
6. $2a, b-a$.
7. a, b.
8. $2, \frac{47}{9}$.
9. $9, -3\frac{3}{5}$.
10. $a, \dfrac{b^2-a^2}{a}$.

Exercise 29.

1. $3, 27, 1, 29$.
2. $5, 21$.
3. $-5, 0$.
4. $1, \frac{1}{2}$.
5. $2, 3\frac{1}{2}$.
6. $\sqrt[3]{5}, \sqrt[3]{-3}$.
7. $a, 0$.
8. $\pm 2c\sqrt{-2}, \pm c\sqrt{-15}$.

Exercise 30.

1. $1, -\frac{1}{2}, -3$.
2. $\dfrac{\pm\sqrt{3}}{2}, \dfrac{1\pm\sqrt{7}}{4}$.
3. $0, \pm a\sqrt{-2}$.
4. $1, 2, \frac{2}{3}$.
5. $1, -1, \frac{2}{7}$.
6. $1, \dfrac{7\pm\sqrt{-59}}{3}$.